A Journey through the World of Combinatorics

Yu-Hong Alexander Wang

Alex's World of Math

A Journey through the World of Combinatorics

By Yu-Hong A. Wang

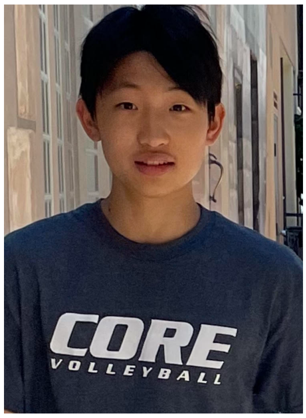
A book about the power of counting the world around us.

Alex's World of Math

About Alex's World of Math

Alex's World of Math is an organization built for the purpose of expanding free educational resources to inspire passion for mathematics among people of all ages. Mathematics is a powerful tool for uncovering patterns in the complex world. Whether it be for high school, college, or independent exploration, Alex's World of Math hopes that its students will gain access to these tools and build them into an integral part of their life and further applicable in the real world. Other resources are available at our website, https://alexsworldofmath.org, and our YouTube channel, https://youtube.com/@alexsworldofmath.

About the Author & Motivation



Yu-Hong Alexander Wang, the author of this book and the owner of Alex's World of Math, is currently a 16-year-old senior enrolled in education from Central Jersey College Prep. Alexander has always loved the art that math provides for him, even back in a private, advanced elementary school, Elite Preparatory Academy, when he learned about slopes and other pre-algebra content in 4th grade. After his transfer to Central Jersey College Prep, the content he learned was more in-depth but not as advanced as before. By 8th grade, he'd been bored of having to re-"learn" pre-algebra. Through Khan Academy, he was able to grasp all of Algebra 1, Geometry, Algebra 2, Pre-Calculus, and one unit of Calculus by the end of 8th grade. After receiving the unfortunate news that he could not take any higher math classes throughout high school due to logistical reasons, he kept pursuit of his passion, exploring outside-of-school options: he took classes at Mathnasium and Center of Talented Youth to refine his knowledge and introduce himself to competition math. Though he was unable to learn anything in all four years of high school math classes, he savored his time outside of school. Self studying Calculus AB, excelling many math competitions and tournaments, dedication for tutoring other students, and joining the Mustang Math community became apart of his normal routine in which he deeply enjoyed.

Motivated to inspire other students just as Sal Khan, the owner of Khan Academy, did, he went out and discovered his passion for combinatorics and discrete math through the AwesomeMath summer program. Using YouTube, college notes, books, and other online resources, what started as writing a short blog on generating functions turned into Alexander's determination to write his own book and share with other students his joyful experiences dealing with combinatorics and discrete math as much as, or even more than, he did. Alexander highly encourages independent studies and exploration of any passion despite the course of any limitations, and believes surpassing them is key to success in the innovative world. Alexander understands the loneliness that comes with independent studies of mathematics passion, and thus, this book aims to provide all passionate—and discouraged—students with something that both allows them to continue exploring their passion, and accompanies them through their journey. The author hopes that this book will be a sustainable, temporary (or perhaps permanent) pathway for readers to reach people or communities with similar passions and are equally willing to nurture their passion alongside them.

Alexander also has collected his experiences with machine learning and AI through online programs and wrote research proposals, literature reviews, and independent research papers on his machine learning models' performances across applicable settings such as predicting heart disease or student GPA—many of which can be seen on the website https://alexsworldofmath.org.

Overview of Content

To make full use of this book, it is advised to attempt all the challenging exercise problems that are provided at the end of each chapter. All sections can be found in the table of contents page. Example problems in each section are walked through step-by-step and proofs are written as encouragement for development of proof-style writing, with many practical and proof exercise problems targeting those skills. Additionally, important theorems and formulae are provided.

This book contains six chapters with in-depth content. Chapter 1 explores introductory principles of counting such as counting permutations and combinations; Chapter 2 introduces the binomial theorem and the important proof method of double counting; Chapter 3 dives deeper into complex counting techniques such as the Principle of Inclusion-Exclusion and Bijections; Chapter 4 is on a powerful method called generating functions that provide another way to count by encoding and decoding a sequence into a power series; Chapter 5 discusses an author-favorite topic, recurrence relations and some of its real-life applications; and finally, Chapter 6 opens the door for the author's #1 favorite topic and one of the most prominent and important fields of combinatorics, partition theory. Any corrections on errors or suggestions are welcomed—please contact us through alexanderwang3445@gmail.com.

Acknowledgements

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1. At the top, there are those who are born with talents and resources, and those with scarce resources who work twice as hard.

2. “Talent is important, but how one develops and nurtures it is even more so.”
- Terence Tao, mathematician.

Contents

CHAPTER 1

Foundations of Counting

1.1 Important Notations

Notation	Description	Example
$\therefore$	"Therefore"	$1+1=2,\ 2+3=5,$ By substitution, $1+1+3=5$.
$\forall$	"For all"	Let $f(x)=3\ \forall x>5$
$\exists$	"Exists" or "There exists"	Let integer $a=3$ and $\exists$ integer b in which $b>15$
$\square$	The proof is finished. Abbreviation in Latin: "quod erat demonstrandum"	Prove that when $ab=2$ and a,b are positive integers, then there are only two solutions to this equation. The only possible integer factor pair that multiplies to 2 is Therefore, the only combinations of (a,b) possible that satisfies

Notation	Description	Example
		the conditions is $(1,2)$. $\square$
$\in$	An element is "in" a set.	$2 \in \{1,2,3\}$
$\cup$	Union of two sets, or the set of elements in either or both sets.	$\{1,2\} \cup \{2,3\} = \{1,2,3\}$
$\cap$	Intersection of two sets, or the set of elements that is part of both sets.	$\{1,2\} \cap \{2,3\} = \{2\}$
$\subseteq$	A set is a subset of another set.	$\{1,2\} \subseteq \{1,2,3\}$
$\subsetneq$	A set is a proper/strict subset of another set (i.e., a set that is not equal to the set it is a subset of)	$\{1\} \subsetneq \{1,2\}$
$\lvert\cdot\rvert$	Cardinality, or the size, of a set	$\lvert\{1,2,2\}\rvert= 3, \lvert\{3,4,9,2\}\rvert= 4$
$\emptyset$	The empty set, a set with no elements.	$\emptyset$ or $\{\}$
$\mathbb{N}$	The set of natural numbers.	$\mathbb{N} = \{1,2,3,\ldots\}$
$\mathbb{Z}$	The set of all integers ($\mathbb{Z}^+$ and $\mathbb{Z}^-$ denote positive and negative integers, respectively).	$\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$
$\mathbb{Q}$	The set of rational numbers.	$\mathbb{Q} = \left\{\frac{p}{q} \mid p,q \in \mathbb{Z}, q \neq 0\right\}$
$\mathbb{R}$	The set of real numbers.	$\mathbb{R}$
$\mathbb{C}$	The set of complex numbers (includes real and imaginary numbers).	$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$
$\{x \mid \text{condition } y\}$	Set builder notation, representing numbers x (e.g., x = all real numbers) that satisfies y: "x such that condition y".	$\{x \in \mathbb{R} \mid x > 100\}$
$P(n,k)$	Permutations of k elements from a pool of n elements, or arrangements with regard to order.	$P(5,2) = 20$
$C(n,k) = \binom{n}{k}$	Combinations (arrangements without regard to order) of choosing k elements from a pool of n elements, which is also the binomial coefficient	$\binom{5}{2} = 10$
$\prod_{i=1}^{n} a_i$	Product of a sequence of terms a_i from $i = 1$ to n.	$\prod_{i=1}^{3} i = 1 \times 2 \times 3 = 6$
$\sum_{i=1}^{n} a_i$	Sum of a sequence of terms a_i from $i = 1$ to n.	$\sum_{i=1}^{3} i = 1+2+3 = 6$
$f : A \to B$	A function f mapping set A to set B.	$f(x) = x^2$ for $x \in \mathbb{R}$

1.2 Introduction to Combinatorics

Let's begin our journey into the fascinating world of combinatorics with some basic counting principles. As the name suggests, combinatorics indeed studies methods of counting the ways something can be done. However, it is more challenging than it sounds.

Essentially, combinatorics studies determining outcomes of some event with parameters that are in the form of a set, or multiple sets that share common characteristics but often with different sizes. One of the most common examples are partitions of a number, which we will discuss in a later chapter. Instead, let's take a look at the classic example of rolling dice.

Example 1.2.1

How many outcomes are there for rolling one die? How about two dice? 3 dice?

Solution. Notice that each die contributes 6 "s." Then, we can generalize the number of possible outcomes of rolling any n dice, $6n$. Therefore, one die obtains 6 outcomes, two dice obtains 12, and 3 dice obtains 18. †

Please note that outcome is controlled by constraints imposed by the problem itself. One example of that is whether we define two events as mutually exclusive or not. In the case of rolling two dice, and we are looking to find the number of ways to roll a sum of n, each sum are obviously mutually exclusive—that is, the events cannot happen in the same time. For instance, rolling a sum of 3 is mutually exclusive with rolling a sum of 4, because we cannot roll a sum of 3 and 4 at the same time. An important property to consider when events are mutually exclusive is the **addition principle**.

Definition 1.2.2 (Addition Principle)

The addition principle states that if two sets of events, A and B, are disjoint (that is, the intersection of the sets is the empty set of size 0: $A \cap B = \emptyset$), then the union of the two sets is the size (known as "cardinality") of each added together, denoted by

$$|A \cup B| = |A| + |B|.$$

You may think of this as another way of saying that two sets that do not share any common elements have the union of the sum of each set's size. Let's represent this with a visualization of this:

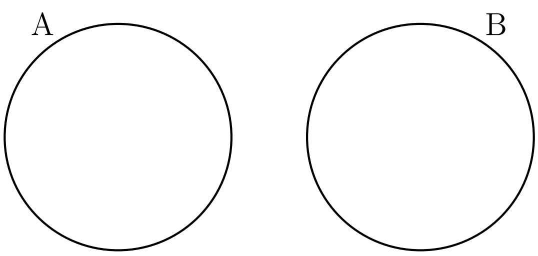


Figure 1.2.1: Mutually Exclusive: No Intersection

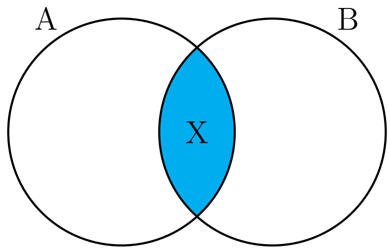


Figure 1.2.2: Not Mutually Exclusive: Intersection "X"

A useful principle will be discussed later on that will help determine the union of sets that are not mutually exclusive, helping account for the overlap of the area of the two sets if they do have any intersection.

Let's return to the dice examples. Using two dice, the number of ways to roll a sum of 3 (A) is 2, and the number of ways to roll a sum of 4 (B) is 3. Since we know we cannot obtain the two events at the same time, or mutually exclusive, then we know the number of ways to roll either a sum of 3 or 4, or the union of the two sets A and B, is $2 + 3 = 5$.

But how do we know the size of A and B are 2 and 3, respectively? Well, in combinatorics, we very often break problems into smaller pieces in order to generalize the thing we wanted to investigate. This is a great example of that. Let n be the sum of rolling two dice, d_1, d_2. We might first notice that n has a minimum of 2 (rolling the lowest possible number, 1, on both d_1 and d_2) and a maximum of 12 (rolling the highest possible number, 6, on both d_1 and d_2).

Let's represent each possible sum or combination of rolls using (number rolled by d_1, number rolled by d_2). Then, you might notice that each combination is unique when trying to achieve a sum of n, where the number rolled by d_2 is forced by the result we got on d_1. Say we wanted to roll a sum of 3, then we know that if d_1 rolls 1, then d_2 must roll 2. Similarly, if d_1 rolls 2, then d_2 must roll 1 in order to achieve the sum of 3.

Therefore, there must only be 0 or 1 way of rolling a 3, or any n, when d_1 has already been rolled. We will apply the casework technique here, where we investigate each case. For the case of achieving a sum of $n = 2$, d_1 can only be 1, and therefore there is only one case that can work $\to (1, 1)$. For $n = 3$, d_1 can either be 1 or 2, and so there are only two cases that work $\to (1, 2), (2, 1)$. For $n = 4$, d_1 can be either $1, 2$, or 3, and therefore there are three cases that work $\to (1, 3), (2, 2), (3, 1)$. You may now see the pattern, and this pattern of having $n - 1$ ways of achieving a sum of n goes up to $n = 7$, having 6 possible ways. At $n = 8$, our range of options for d_1 shrinks due to the 1 not being possible anymore, as no number rolled by d_2 can complement d_1 rolling 1 to get the number 8.

Additionally, $n - 1$, or 7, is unable to be rolled as well. Therefore, d_1 can only roll numbers from $2 - 6$ to achieve $n = 8$, which is 5 ways $\to (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)$. For rolling $n = 9$, there are only feasible options for d_1 from $3 - 6$, and so there are 4 ways $\to (3, 6), (4, 5), (5, 4), (6, 3)$. This pattern continues until we reach $n = 12$, in which there is only one way to roll $\to (6, 6)$. Let's represent this in a table:

n	**Total number of ways to roll using d_1, d_2**	**x_n, the set of all possible combinations of d_1, d_2 to roll n**
2	1	(1, 1)
3	2	(1, 2), (2, 1)
4	3	(1, 3), (2, 2), (3, 1)
5	4	(1, 4), (2, 3), (3, 2), (4, 1)
6	5	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)
7	6	(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)
8	5	(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)
9	4	(3, 6), (4, 5), (5, 4), (6, 3)
10	3	(4, 6), (5, 5), (6, 4)
11	2	(5, 6), (6, 5)
12	1	(6, 6)

Table 1.2.1: Total number of ways to roll using two dice

Looks pretty cool right? The first few rows of Pascal's triangle is hidden in here. To explain why this somewhat captures features from Pascal's triangle, it is because the first number in each parenthesis increase while the second number decreases by 1 every index we move. Do not fret if you are not familiar with the Pascal's triangle, we will go over it in a similar chapter where we will see a similar increase/decrease in numbers in the powers of binomial coefficient sums.

Remember that each set x_n to obtain a different sum n is mutually exclusive to one another. We can see that here, where no sets share a common element. Let's check if the addition principle works here then, and we know that there are 36 total unique ways of rolling 2 dice.

Proof. By the addition principle, since all of these events for each n represented by (pairwise) disjoint sets x_n are mutually exclusive, then the sum of sizes of each set x_n, which are also written in the second column of the table, must also equal to the union of all sets x_n or the total number of ways of rolling any n (36). That is,

$$\sum_{n=2}^{12} |x_n| = x_1 \cup x_2 \cup \ldots \cup x_{12} = \boxed{36}.$$

Let's determine if this is true by adding up all the numbers in the second column.

Indeed,

$$\begin{aligned}1+2+3+4+5+6+5+4+3+2+1 &= \frac{6(6+1)}{2} + \frac{5(5+1)}{2} \\ &= \frac{6(7)+5(6)}{2} \\ &= \frac{42+30}{2} \\ &= \frac{72}{2}\end{aligned}$$

$$= \boxed{36} \checkmark$$

(using the sum of arithmetic sequence formula, $\frac{n(n+1)}{2}$). Note that the addition principle may be easily proved with a proof by induction (this proof technique will be explored later as well). □

But how did we know that the total number of ways of rolling dices were 36? We can again think of the first dice d_1 having 6 options and for each of the 6 options, there are 6 options that we can pick for the second dice d_2. In other words, we get $6 \cdot 6 = 36$.

Example 1.2.3

What if we added a restriction that said we cannot have the number 1 appear on the second dice? That is, the second dice only ranges from $2 - 6$ now.

We can use complementary counting by counting the total number of possibilities subtracted by the ways we cannot order it, which would be $36 - 6$, as there are 1 appears 1 time per number of the first one, making it 6 times. This leaves us with 30 unique combinations of the numbers on two dices where the second dice is not 1. But we can also think of this another way: for every number chosen on dice 1, there are 5 possible numbers that dice 2 can roll that satisfies the condition. That is, $6 \cdot 5$ ways of doing so. Obtaining 36 through this process, complementary counting, and also by the addition principle provide ways for double counting, where we prove an equation by counting each side of the equation with true methods of the same problem. We will dive into this more in a later chapter.

Notice that we had $6 \cdot 6$ before and now $6 \cdot 5$ based on the initial conditions. Well, this is called the **multiplication principle**.

Definition 1.2.4 (Multiplication Principle)

If n events each have x_i possible unique outcomes, for $i = 1, 2, 3, \dots, n$, where each x_i is not affected by the other outcome (i.e. they are pairwise independent), then the total possible outcomes is denoted by

$$\prod_{i=1}^{n} x_i.$$

This is the basis of combinatorics, as it will be highly useful in computing combinations and permutations among many other things.

A simple demonstration of this would be choosing which shirt or pants to wear, when you have 2 options of it each. This is $2 \cdot 2 = 4$ total outfits that can be made:

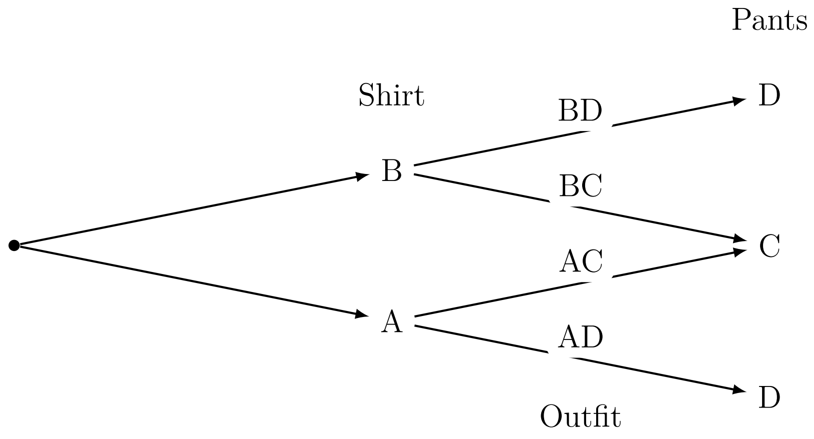


Figure 1.2.3: Multiplication Principle Represented by a Tree

Now that we understand these principles, let's move on to arrangements and orderings after some exercises.

Youtube Lectures 1.2

1. The Addition Principle - Counting Principles
2. The Multiplication Principle - Counting Principles

Exercises 1.2

Exercise 1.2.1

How many outcomes are there when rolling two dice where the sum is exactly 5? How about rolling a sum of 8? How about either sum of 5 or 8?

Exercise 1.2.2

A restaurant offers 3 types of sandwiches and 4 types of drinks. How many different meal combinations can you create? If a new rule is introduced where each meal can include only one type of sandwich and one type of drink, but you must avoid a specific combination (e.g., sandwich A with drink 2), how many different meal combinations can you now create?

Exercise 1.2.3

A student can choose one book from the school library or one from the city library. If the school library has 10 books and the city library has 15 books, how many total choices does the student have?

Exercise 1.2.4

Suppose three cards are chosen at random from a standard 52-card deck, with replacement. How many sequences of three cards are there, where all three cards are from the same suit? How about without replacement?

Exercise 1.2.5

When rolling two dice, how many ways can you get an even sum? How about odd? How many ways can you get a sum greater than or equal to 8 if the two dice are now indistinguishable?

Exercise 1.2.6

Event A can be expressed by set A. Event B can be expressed by set B. These events are not mutually exclusive: $|A \cap B| = 15$. Additionally the cardinalities of both sets are given: $|A| = 42, |B| = 36$ What is the union of Event A and Event B ($|A \cup B|$)?

Exercise 1.2.7

How many ways are there to arrange 3 distinct cookies into 3 spots? Assign each cookie a label and apply casework.

1.3 Permutations and Combinations

Permutations and combinations are powerful tools in combinatorics and are two of the foundational principles required to solve higher level problems. Let's dive right in.

Example 1.3.1

Suppose there are 5 different color beads labeled

$$b_1, b_2, \ldots, b_5$$

and say we wanted to create strings of 3 unique-colored beads to later connect into a bracelets. How many outcomes or strings are possible to be made? That is, how many ways is it possible to have a selection of 3 beads without repetition from 5 uniquely labeled beads where order matters?

Solution. It is common to think about these types of combinatorial problems like placing x amount of items from a pool of y items into x "slots". In this case, we can taking 3 items from a pool of 5 items and seeing how many times we can place them into 3 slots. We can then generalize the amount of ways of satisfying the problem for each slot.

$$b_1, b_2, b_3, b_4, b_5$$

(5 items)

$$\underset{\text{slot 1}}{\underline{\quad ? \quad}} \quad \underset{\text{slot 2}}{\underline{\quad ? \quad}} \quad \underset{\text{slot 3}}{\underline{\quad ? \quad}}$$

Let's start with slot 1, which would be our "first pick". How many different beads can be selected from the pool s.t. ("such that") we are still satisfying the condition of having 3 unique beads? Well, if this is the first pick then slot 1 has 5 possible beads that it could take.

$$\underset{\text{slot 1}}{\underline{\quad 5 \quad}} \quad \underset{\text{slot 2}}{\underline{\quad ? \quad}} \quad \underset{\text{slot 3}}{\underline{\quad ? \quad}}$$

Next, slot 2 is our second pick. Note that we already picked one of the beads, and since all 3 beads we select must be unique, we cannot pick the same bead that we picked for slot 1. Therefore, slot 2 can only take 4 different beads.

$$\underset{\text{slot 1}}{\underline{\quad 5 \quad}} \quad \underset{\text{slot 2}}{\underline{\quad 4 \quad}} \quad \underset{\text{slot 3}}{\underline{\quad ? \quad}}$$

Lastly, we use the same logic. We have already selected 2 of the 5 unique beads, so there are only 3 left to pick from.

$$\underset{\text{slot 1}}{\underline{\quad 5 \quad}} \quad \underset{\text{slot 2}}{\underline{\quad 4 \quad}} \quad \underset{\text{slot 3}}{\underline{\quad 3 \quad}}$$

This representation of the problem allows us to think of it like this: For each and any of the 5 beads we pick for slot 1, we have 4 choices for slot 2. For each of the choices in slot 2, we have 3 beads possible to pick from in order to form 3 unique beads across the 3 slots. Then, it becomes apparent that this is a situation where the multiplication principle applies.

Indeed, the number in each slot represent the possible beads of selection, which are all independent from one another.

The independence comes from the fact that in slot 1, there can be any of the 5 beads, and in slot 2, there can be any of the 4 beads left over, and same goes for slot 3. No outcome of any slot affect the outcome of another slot. This means no matter which bead is selected to slot 1, slot 2 will indefinitely have 4 beads to pick from to keep the distinctness of the 3 selections. Therefore, there are

$$\underset{\text{slot 1}}{\underline{\quad 5 \quad}} \cdot \underset{\text{slot 2}}{\underline{\quad 4 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 3 \quad}} \quad = \boxed{60}$$

possible outcomes (or possible ways of selecting 3 distinct beads). †

Example 1.3.2

In an animal shelter, there are n kittens, each with a unique tag numbered from 1 to n. You want to select k kittens and give each one a unique bow from a set of k distinct bows. How many different ways can you choose and arrange the kittens such that each receives a distinct bow?

Solution. Instead of thinking of this problem as two parts, we can think of it as one process. That is, treating each bow as a "slot." So then, the question can be reframed as the number of different ways to place n distinct kittens into k distinct slots ("ordered sequence of kittens"). Then, this is exactly the same as the previous problem dealing with beads. So if we use our slots method, then we are able to see that the first selection of any kitten into the first bow has n possible kittens and the second pick, in order to keep unique orderings (no repetition), there are only $n-1$ choices left. The pattern keeps following until we reach the last, kth bow where there are $n-k+1$ kittens left to choose:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+2) \cdot (n-k+1)$$

total number of ways to arrange.

Notice that if k was equal to n, then we would have

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1 = n!$$

total ways to arrange.

This case should allow us to work a formula for any n and any k to answer the problem. Let's try to devise a relationship from this case and other cases below it, as we must remember that n is the highest number that k is able to take.

Let us try $k = n-1$, then we would have:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-(n-1)+2) \cdot (n-(n-1)+1)$$

$$= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2$$

total ways of arranging.

Do you see what is happening here? Let's try $n-2$:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-(n-2+2) \cdot (n-(n-2)+1)$$

$$= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 4 \cdot 3$$

total ways of arranging.

Ah, it's clear now. Each time we subtract one from k when it starts at $k = n$, one starting number from $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ gets booted out. That is, $k = n-c$ would have the first c terms from $n!$ removed. But if we wanted to devise a formula, we need only two variables. What if we said instead of terms being removed, the amount of terms of $n!$ are kept. Then, that will be the last $n-c$ terms of $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

Wait! We also know that $n - c = k$. Great! So this also means that the last k terms of $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ are kept, or in other words, those terms multiply to become the total ways of arranging the kittens.

We are very close now! Those k terms are $n, n-1, n-2, \ldots, n-k+1$ as we wrote before. To achieve this, the other $n-k$ terms, $n-k, n-k-1, n-k-2, \ldots, 3, 2, 1$, are to be removed from $n!$. How do we remove numbers from $n!$? We divide. We want to remove $n-k$ starting terms of $n!$, which is the product

$$(n-k) \cdot n-k-1 \cdot n-k-2 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = (n-k)!\,.$$

There we have it, the total number of ways to choose and arrange n kittens s.t. each kitten receives one distinct bow is equal to:

$$\frac{n!}{(n-k)!}$$

†

This is known as permutations.

Definition 1.3.3 (Permutations)

Permutations are ways to arrange k items from a pool of n distinguishable (or "distinct") items into an ordered sequence. Or in other words, this is the number of ways to arrange those k items into a sequence where *order matters and repetition is not allowed*. Permutations are denoted by $P(n, k)$, where:

$$\begin{aligned} P(n,k) &= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+2) \cdot (n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

You may also see this same process be expressed differently, such as Example 1.2.3 shown below.

Example 1.3.4

You have 8 different books, and you want to arrange 4 of them on a shelf. How many different ways can you arrange these 4 books?

Solution. This is exactly what we had before, so we know that we are trying to choose 4 distinct items from a set of 8 books. This is 4 slots, and so we have the first choice being 8, then 7, then 6, then for the last slot, there are 5 choices. We can also apply our formula to get the same thing:

$$\begin{aligned} P(8,4) &= \frac{8!}{(8-4)!} \\ &= \frac{8!}{4!} \end{aligned}$$

$$= 8 \cdot 7 \cdot 6 \cdot 5$$
$$= \boxed{1680}$$

total ways to arrange 4 distinct books on the shelf. †

Example 1.3.5

What if you still have 8 different books and you still want to choose 4 of them to place on a shelf. Now, how many different groups of 4 books can you select? In other words, the ordering does not matter.

The wording on these problems are very important. When "groups" or "subsets" are mentioned, they imply a property in which however you switch the order of the chosen sequence, it is all counted as one arrangement.

In this case, we have 8 books that we want to count the arrangement of into 4 slots, but just removing the cases in which the same group is repeated. To do this, we can again use our permutation principles and divide by some number of times that the sequence is repeated. So this would be again:

$$\frac{8}{\text{slot 1}} \quad \frac{7}{\text{slot 2}} \quad \frac{6}{\text{slot 3}} \quad \frac{5}{\text{slot 4}}$$

which is

$$8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

total arrangements.

Now we need to look at each specific sequence. Let's label each book by $b_1, b_2, \ldots, b_8$, then one of the arrangements can be:

$$(b_1, b_7, b_3, b_4).$$

How many times can these four distinct books be arranged into the four slots, including the arrangement mentioned above? Well, this is just another permutation, right? This is repetition not allowed, since they are distinct books, and it is also order matters as we want to count how many times it can be distinctly ordered with these same four books. To this extent, we have:

$$\frac{4}{\text{slot 1}} \quad \frac{3}{\text{slot 2}} \quad \frac{2}{\text{slot 3}} \quad \frac{1}{\text{slot 4}}$$

amount of choices per slot, and by the multiplication principle, we have

$$4 \cdot 3 \cdot 2 \cdot 1 = 24$$

total arrangement per sequence of four.

This can also be obtained by our permutation formula:

$$P(4, 4) = \frac{4!}{(4-4)!}$$

$$= \frac{4!}{0!}$$
$$= \frac{24}{1}$$
$$= 24.$$

So if each group of four books can be arranged 24 ways, and we are looking for the number of groupings possible, then we may utilize this equation, given by the multiplication principle and using our answer of total arrangements from earlier:

$$\text{\# of groupings } \times 24 = 1680$$

That means the total number of groupings in which order does not matter is

$$\frac{1680}{24} = 70.$$

Let's try to generalize this. What if we had instead of 8 total distinct books, we had n distinct books to choose from and we are asked to find the total number of subsets of size k.

We can again use our equation to find a general expression for the number of groupings:

$$(\text{\# of groupings}) \times (\text{\# of possible arrangements per group of size } k)$$
$$= \text{ total \# of arrangements of size } k$$

We know that the # of possible arrangements per group of size k is $P(k,k)$, and that the total # of arrangements of size k is $P(n,k)$. So,

$$(\text{\# of groupings }) \times P(k,k) = P(n,k)$$

We also know from Definition 3.3 that

$$P(k,k) = \frac{k!}{(k-k)!} = \frac{k!}{0!} = \frac{k!}{1} = k!$$

and

$$P(n,k) = \frac{n!}{(n-k)!}$$

Substituting back into the equation, we get:

$$(\text{\# of groupings }) \times k! = \frac{n!}{(n-k)!}$$

$\therefore$ ("therefore") The total # of groupings of size k chosen from a set of n distinct items is

$$\boxed{\frac{n!}{(n-k)!\,k!}}$$

Definition 1.3.6 (Combinations)

Combinations is the practice of "choosing" k items from a set of n items without regard to order. Combinations can be generally expressed as $C(n, k)$ ("n choose k"), where

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Note that $C(n, k) \times k! = P(n, k)$, and that $\binom{n}{k}$ is another notation called the binomial coefficient that may be used for $C(n, k)$ (this will be discussed later on).

Example 1.3.7

How many subsets of size 5 can be formed from a set of cardinality ("size") 9?

Hint: Repetition is not allowed in sets, so all items in a set are always distinct by definition.

Solution. Again, if we are trying to form subsets of size 5, then we are looking for unordered sequences, or groupings, since any set or subset are considered the same even if their order is switched. Additionally, items in sets are distinct, so this is basically the same as the previous problem where we are looking for groupings. Let's now apply our combinations formula:

$$\begin{aligned} C(9,5) &= \frac{9!}{(9-5)!\,5!} \\ &= \frac{9!}{(4)!\,5!} \end{aligned}$$

Let's separate out 9!:

$$\begin{aligned} \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{4! \cdot 5!} &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{4!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{9 \cdot (4 \cdot 2) \cdot 7 \cdot (3 \cdot 2)}{4 \cdot 3 \cdot 2 \cdot 1} \\ &= 9 \cdot 2 \cdot 7 \\ &= \boxed{126}. \end{aligned}$$

$\therefore$ ("therefore") there are 126 subsets of size 5 possible to be formed by a set of size 9. †

Definition 1.3.8 (Choosing Subsets)

The number of subsets of size k that can be formed by a set of size n (also called an n-set) is given by Definition 3.6:

$$\begin{aligned} C(n,k) &= \frac{P(n,k)}{k!} \\ &= \frac{n!}{(n-k)!\,k!} \\ &= \binom{n}{k} \end{aligned}$$

$\binom{n}{k}$ is the most commonly used notation called the binomial coefficient, and is pronounced "n choose k."

Example 1.3.9

How many total subsets of any size can be formed from a set with n elements?

Solution. Of course, we can use the fact that the # of subsets that can be formed of size 0 is $C(n,0)$, and do the same for all numbers up to n which is all the possible sizes that a subset of the set of size n can take. Then, we can add all of them up and obtain our answer:

$$\sum_{k=0}^{n} C(n,k)$$

Although this way is inefficient, let's test a few values of n to see where this might be converging to, then explore an easier interpretation.

What if $n=0$? Then we can only have a subset of size 0:

$$C(0,0) = \frac{0!}{(0-0)!\,0!} = \frac{1}{(1)1} = 1.$$

What if $n=1$? Then we can have a subset of size 0 and a subset of size 1:

$$C(1,0) = \frac{1!}{(1-0)!\,0!} = \frac{1}{(1)1} = 1,$$

$$C(1,1) = \frac{1!}{(1-1)!\,1!} = \frac{1}{(1)1} = 1.$$

Adding these together, we have $1+1=2$ subsets that can be formed.

What if $n=2$? Then we can have a subset of size 0, 1, and 2:

$$C(2,0) = \frac{2!}{(2-0)!\,0!} = \frac{2}{(2)1} = 1,$$

$$C(2,1) = \frac{2!}{(2-1)!\,1!} = \frac{2}{(1)1} = 2,$$

$$C(2,2) = \frac{2!}{(2-2)!\,2!} = \frac{2}{(1)2} = 1.$$

Adding these together, we have $1 + 2 + 1 = 4$ subsets that can be formed.

What if $n = 3$? Then we can have a subset of size 0, 1, 2, and 3:

$$C(3,0) = \frac{3!}{(3-0)!\,0!} = \frac{6}{(3)1} = 1,$$

$$C(3,1) = \frac{3!}{(3-1)!\,1!} = \frac{6}{(2)1} = 3,$$

$$C(3,2) = \frac{3!}{(3-2)!\,2!} = \frac{6}{(1)2} = 3,$$

$$C(3,3) = \frac{3!}{(3-3)!\,3!} = \frac{6}{(1)6} = 1.$$

Adding these together, we have $1 + 3 + 3 + 1 = 8$ subsets that can be formed.

This is very interesting, it seems like this is following the path of 2^n. We can also tell that this is following the binomial coefficient, pattern, which is denoted by $\binom{n}{k}$ and is equal to $C(n,k)$. We will go through this in a later chapter, but it is quite interesting to see them appear here.

Anyway, there is a better way of interpreting this problem. Instead of thinking of individual subset sizes, we can use the fact that subsets that are smaller than the original set are the same as the original set with some items not appearing.

For example, if we have the set: $\{3, 5, 8\}$, we can think of it having 3 spots with an element either being present or not. An example subset is $\{5, 8\}$, which can be represented by $\{0, 1, 1\}$, using binary of 0 to represent an element being absent from the set and 1 being present.

In this case, the middle 1 would represent 5 being present, while the right 1 would represent 8 being present. Basically, for each element, we are choosing from the set $\{0, 1\}$, so we are making n selections of either 0 or 1.

Thinking in this way allows us to realize that this successfully captures all possible subsets in one order, which makes it easier for us as we would not have to try to remove all possible orderings per group.

Therefore, the amount of choices that can be made can be formed through the multiplication principle, where each element on the subset is binary, and therefore has two possible choices. Since there are n elements, there can be n choices of 2, or $\boxed{2^n}$ many subsets of any possible size able to be formed. †

Example 1.3.10

You have a combination lock that uses n different digits (from 1 to n), and the lock requires a sequence of k digits to open. That is, the digits in the sequence can be repeated, and the order of the digits matters. How many different combinations of digits can you use to set the lock?

Solution. This is essentially asking how many ways to choose k elements from $\{1, 2, 3, \ldots, n\}$ where we form ordered sequences and repetition is allowed. In the previous problem, we actually were choosing n elements from the set $\{0, 1\}$, meaning there were n choices of either 0 or 1.

In this problem, we are looking to select k digits ranging from 1 to n. Therefore, we have n choices k times. Through the multiplication principle, we have that this is just n^k combinations of digits to set the lock. †

Youtube Lectures 1.3

1. Permutations - Counting Principles

2. Combinations - Counting Principles

3. Ordered Selection, Repetition Allowed - Counting Principles

Exercises 1.3

Exercise 1.3.1

How many different ways can you arrange the letters in the word "MATH"? How about "BOOK"?

Exercise 1.3.2

A lottery requires you to pick 6 numbers out of a pool of 49. How many unique combinations of numbers can you select?

Exercise 1.3.3

In a video game, a player can choose from five distinct power-ups each turn. Over 6 turns, how many different sequences of power-ups can be chosen if repetition is allowed?

Exercise 1.3.4

In poker, how many different ways can you form a hand that includes two cards of one rank and three cards of another rank?

Exercise 1.3.5

Twelve people, consisting of six men and six women, are seated around a circular table so that men and women alternate seats. Chairs do not matter and only who is next to whom matters. If left and right are distinguishable, how many different seating arrangements are possible? What if instead, eight friends are sitting around the same circular table under the same conditions, but two specific individuals, X and Y, must not sit next to each other. Then, how many different seating arrangements are possible?

Exercise 1.3.6

A "legal" sequence of parentheses is one where each opening parenthesis has a matching closing parenthesis. Show that the number of legal sequences of length $2n$ is given by the formula

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

Exercise 1.3.7

How many ways can eight indistinguishable rooks be placed on a chessboard so that no two attack each other (in other words, they cannot be aligned by any rows or columns of squares)?

Exercise 1.3.8

Nine people sit down for dinner and there are three meals to pick from. Three people order the vegetarian meal, three order the chicken salad meal, and three order the hot pot meal. How many ways can the waiter serve these meal types to the nine people such that there is exactly one person who receives the type of meal they ordered?

1.4 Stars and Bars

Now, let's recall the definition of combinations and permutations. Say we are choosing k elements from the set $\{1, 2, \ldots, n\}$. If we wanted to form ordered sequences with all choices or terms in that sequence being distinct, then the number of ways to do that is a permutation, denoted by $P(n,k)$. If we wanted to form an unordered sequence with all choices or terms being distinct, then the explicit formula for counting this would be denoted by $C(n,k)$.

In addition, we have just discovered that when we want to count the number of ordered sequences that can be formed with repetition allowed is n^k. Let's create a table:

	Ordered Sequence	Unordered Sequence
Repetition Allowed	n^k	?
Distinct Choices	$P(n,k)$	$C(n,k)$

Table 1.4.1: Choosing k from $\{1,2,\ldots,n\}$

Now, what is the formula for the last box? Note that this number of ways for this box will indeed be less than n^k because we are finding unordered sequences rather than ordered.

You may be first tempted to try $\frac{n^k}{k!}$, based on how we achieved $C(n,k)$ by dividing $k!$ per choice. However, we are unable to do this because each option is no longer distinct, so they can be anywhere and can appear anywhere from 0 to n times. Here is a counterexample setting $n=3, k=4$:

$$\begin{aligned}\frac{n^k}{k!} &= \frac{3^4}{4!}\\ &= \frac{81}{24}\\ &= \frac{27}{8},\end{aligned}$$

which is not an integer and it is impossible to arrange in non-integer ways.

So how do we even go about doing this? One way to do this is to assign multiplicities to each choice when choosing k objects from a set of n with repetition allowed and disregarding order.

Multiplicities is defined by the following: If an object from the original set of size n is chosen 3 times, it will have a multiplicity of 3. If it is chosen 0 times, the multiplicity will be 0. One of these multiplicities have a minimum of 0, denoting it not being chosen, and maximum of k, denoting that it is chosen all k times. Let's denote the multiplicity for each object i to be k_i, in which $1 \le i \le n$.

If these multiplicities assigned are all either 0 or 1, then we would be dealing with all distinct choices, which fall into the combinations category. Otherwise, if they were > 1 then we would have repeated choices. We need to be able to account for all the possibilities. This may seem daunting but it can actually be solved in a very simple way.

Remember when we defined k_i to be multiplicities of each object i, meaning how many times each object appears? Well, if that is the case, then we also know that all the objects' multiplicities added together equals to the amount of objects being chosen, which is also just the original k itself;

$$\sum_{i=1}^{n} k_i = k_1 + k_2 + \cdots + k_n = k$$

(there are n objects, and each i are just labels to distinguish each object).

Please think of counting this kind of situation with repetition allowed and having an unordered selection being choosing indistinguishable k objects into n distinguishable (since items in sets are distinct, and we were choosing originally from a set of n distinct items) bins or slots.

So we have

$$k_1 + k_2 + \cdots + k_n = k,$$

where $k_i \geq 0$. The key is to notice that the number of possible ways to arrange it so that this is true is equal to the number of solutions for this equation. But how do we find the number of solutions? There's a neat trick, where we can think of the equation as separating the number k into bins represented by each object multiplicity k_i.

Essentially, we are separating k into k 1's and counting the amount of ways we can form n groups from those 1's:

$$1 + 1 + 1 + 1 + \cdots + 1 \to k_1 + k_2 + \cdots + k_n.$$

Look at the plus signs, notice that on the left we started with $k-1$ "+" because there were k terms, while on the right, there are $n-1$ "+" as there were n terms, representing n objects from the original set. Remember that these k_i are multiplicities of each object i from the original set of n objects.

Oh! We can think of this grouping as the # of ways of choosing $n-1$ "+" on the right side from the original $k-1$ "+" on the left side. The "+" that we do not choose are evaluated, meaning the numbers to the left and right of the "+" are added together, which forms the k_i groups separated by $n-1$ plus signs. These plus signs essentially are "bars" that separate these groups.

For clarification, let's take the example with $n = 5$, and $k = 3$. So we substitute in the equation

$$k = k_1 + k_2 + \cdots + k_n,$$

and we obtain

$$1 + 1 + 1 = k_1 + k_2 + k_3 + k_4 + k_5.$$

Notice that we cannot combine it the 2 "+" into the 4 "+" so there are 0 ways to do so. This is an error due to us assuming each object's multiplicity group are groups of 1's, which assume that multiplicities of each object $k_i \geq 1$ instead of 0.

We will account for this situation right after we demonstrate this concept of combining 1's. So instead, let's let $k = 5$ and $n = 3$, which means we are choosing an unordered selection of 5 items from $\{1, 2, 3\}$ with repetition allowed. Then this means that we have 3 objects from the original set and so we have:

$$5 = k_1 + k_2 + k_3,$$

or

$$1 + 1 + 1 + 1 + 1 = k_1 + k_2 + k_3.$$

So now we can demonstrate the notion of combining the "+" from the left and we see we have 4 "+" that we want to choose 2 "+" to be left. Let's replace the "+" on the right with bars ("—") to show the separation "+" signs that we would like to achieve:

$$1 + 1 + 1 + 1 + 1 = k_1 | k_2 | k_3.$$

One way to have two separation "+" signs is

$$1 + 1|1 + 1|1 = k_1|k_2|k_3.$$

All we did was choose 2 of the "+" signs on the left to become bars that create three groups that we wanted.

To find how many ways we can do that, it is just counting combinations of 2 bars from a set of 4 "+" signs, which is

$$C(4,2) = \binom{4}{2} = \frac{4!}{(4-2)!\,2!} = \frac{4!}{2! \cdot 2!} = 6.$$

Going back to the general case for any n, k, there are $k-1$ "+" signs on the left and $n-1$ "+" signs on the right. The number of combinations for this is again, the number of "+" signs on the left choosing the number of "+" signs on the right is

$$C(k-1, n-1) = \boxed{\binom{k-1}{n-1}}.$$

Lemma 1.4.1 (Integer Equation Solutions, Part 1)

For any equation $k_1 + k_2 + \cdots + k_n = k$, where $k, k_i \in \mathbb{Z}$ ("are in the set of integers") and that $k_i \geq 1$, the number of solutions is given by

$$\binom{k-1}{n-1}.$$

Now we are only one step away from finding our explicit formula for the number of ways to choose k items from $\{1, 2, \ldots, n\}$ with unordered selection and repetition allowed. Remember that the case we just found was satisfying each group requiring to have at least 1 multiplicity? We want each object in the original set of $\{1, 2, \ldots, n\}$ to have multiplicities of ≥ 0 instead of 1, as we do not always have to choose all objects, as some that has multiplicity of 0 means they are not chosen.

If the equation $k = k_1 + k_2 + \cdots + k_n$ has $\binom{k-1}{n-1}$ solutions where $k_i \geq 1$, then let's try to obtain that same form using what we desire:

$$k = k_1 + k_2 + \cdots + k_n$$

where $k_x \geq 0$.

Notice that $k_i = k_x + 1$, we are basically remove 1 from every single k_i in order to make k_x, which has our desired multiplicity of greater or equal to 0. So then we do $k_x = k_i - 1$ to change every of the k_i to our desired k_x. Substituting, this becomes

$$k = (k_1 - 1) + (k_2 - 1) + \cdots + (k_n - 1),$$

which is also equal to

$$k_1 + k_2 + \cdots + k_n - n.$$

So

$$k = k_1 + k_2 + \cdots + k_n - n,$$

or

$$k + n = k_1 + k_2 + \cdots + k_n$$

for $k_x \geq 0$.

We can now apply the previous formula we got for the number of solutions for this equation, treating $(k+n)$ as the term "k" from before, as there would be $(k+n)-1$ "+" signs on the left choosing $n-1$ "+" signs on the right. Finally, this would just be $(k+n)-1$ choose $n-1$ solutions to the $k_x \geq 0$ equation:

$$C((k+n)-1, n-1) = \binom{(k+n)-1}{n-1}.$$

There is a symmetry for binomial coefficients that

$$\binom{a}{b} = \binom{a}{a-b},$$

meaning that

$$\begin{aligned}\binom{(k+n)-1}{n-1} &= \binom{(k+n)-1}{(k+n)-1-(n-1)} \\ &= \binom{(k+n)-1}{k+n-1-n+1} \\ &= \boxed{\binom{k+n-1}{k}}.\end{aligned}$$

Theorem 1.4.2 (Integer Equation Solutions, Part 2)
For any equation $k_1 + k_2 + \cdots + k_n = k$, where $k, k_i \in \mathbb{Z}$ ("are in the set of integers") and that $k_i \geq 0$, the number of solutions is given by

$$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}.$$

Amazing. What we just did was a derivation of a useful method called stars and bars (with the main result being Theorem 1.4.2).

Definition 1.4.3 (Stars and Bars)

Stars and bars is a combinatorial technique used to count the number of ways to distribute k indistinguishable items into n distinct bins, where repetition of items is allowed. This is equivalent to selecting k items from a set of n distinct objects $\{1, 2, \dots, n\}$ with repetition.

To apply the stars and bars method, represent the items as "stars" and the dividers between bins as "bars." The total number of positions (stars and bars combined) is $k + (n-1)$. The number of ways to distribute the items is given by choosing either the positions for the stars or the positions for the bars from this total, calculated by

$$\binom{k+(n-1)}{k} = \binom{k+(n-1)}{n-1}.$$

This result could also have been obtained by the following reasoning: We can reword the problem from choosing k items from the set of n distinct items to distributing k identical objects into n distinct bins. We again treat k as 1's but let's define each 1 as a star. So we have k stars to be split into n bins, which can be done by setting $n-1$ bars between some two pairs of stars. For example, if we had $k = 5$ stars to be split into $n = 3$ bins, then one way to divide the stars with bars into the 3 bins is this: $**|**|*$.

This would represent the first bin and second bin having two stars while the third bin has only one star. So then in general, there will be k stars and $n-1$ bars, totaling $k + (n-1)$ positions. Then our answer for a repetition allowed and unordered sequence choosing of k from a set of size n would be finding the total amount of ways to arrange these stars and bars because this, unlike our previous method, ensures that these two bars can be anywhere to form any sized bin from 0 to n, which 0 can be done by the bars being next to each other and n sized bin can be achieved by placing the two bars on the two farthest ends (on the far left and far right).

Now, to find the total number of arrangements possible of $k + (n-1)$ positions with k stars and $n-1$ bars, we need to realize that there are only two distinct objects here to be chosen (you can think of this as counting the ways of distributing $k + (n-1)$ distinct objects into two groups of size k and size $n-1$). Therefore, we can either (1) find the total number of arrangements possible and divide by the number of arrangements possible with that grouping similar to how we did with permutations to find combinations, or (2) we can realize that if we find the number of ways to choose positions for the star, then the rest positions do not need to be calculated as they must be bars (this would be the same around if we found all the positions for the bar, then we have also found the positions for the stars).

Let's first try option (2). The number of ways of distributing $k + (n-1)$ distinct objects into a group of size k (representing the stars) would already decide that the rest of the objects left would go into the other group of size $n-1$. Because we are trying to distribute from a number of distinct objects, we can think of this as distributing from a set, as sets have all distinct elements.

So we are trying to find the number of ways to choose k objects from a set of size $k + (n-1)$, which is also saying we are trying to find the number of subsets of size k

that can be formed from a set of size $k + (n-1)$. We have done this before, and the answer is simply $\binom{k+(n-1)}{k}$. If we did this the other way by choosing a subset of size $n-1$, we would have obtained $\binom{k+(n-1)}{n-1}$. These two formulas are indeed equal. We may check using the combinations formula:

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} \text{ (Definition 1.3.2)}$$

So to check if

$$\binom{k+(n-1)}{k} = \binom{k+(n-1)}{n-1},$$

we can plug in the formula for each side:

$$\begin{aligned}\binom{k+(n-1)}{k} &= \frac{(k+(n-1))!}{(k+(n-1)-k)!\,k!}\\ &= \frac{(k+(n-1))!}{(n-1)!\,k!},\end{aligned}$$

$$\begin{aligned}\binom{k+(n-1)}{n-1} &= \frac{(k+(n-1))!}{(k+(n-1)-(n-1))!\,(n-1)!}\\ &= \frac{(k+(n-1))!}{k!\,(n-1)!}.\end{aligned}$$

Therefore, we have

$$\boxed{\frac{(k+(n-1))!}{(n-1)!\,k!} = \frac{(k+(n-1))!}{k!\,(n-1)!}},$$

which is indeed true.

Now let's try option (1). This way is quite straightforward. Let's use the formula similar to the one that we used to find the combinations formula (please note that this is still different than combinations, as we are placing indistinguishable objects into distinct bins instead of placing distinct objects into distinct bins):

$$\begin{aligned}\#\text{ of unordered groupings } &\times \text{ arrangements possible per grouping}\\ &= \text{ total arrangements possible}\end{aligned}$$

So,

$$\#\text{ of unordered groupings } = \frac{\text{total arrangements possible}}{\text{arrangements possible per grouping}}$$

Again, let me remind you that we are finding the total arrangements of the set of $k+(n-1)$ undistinguishable objects into two distinct bins of size k and $n-1$. The total orderings possible of any m objects is defined by permutations of m objects into m slots. So in our case, we have

$$P(k+(n-1), k+(n-1)) = \frac{k+(n-1)}{(k+(n-1)-k+(n-1))!}$$

$$= \frac{(k+(n-1))!}{0!}$$
$$= (k+(n-1))!$$

Now, we need to divide by the total arrangements possible per grouping. Let's take an example grouping, pretending this is k stars and $n-1$ bars: $||*****$. When the total arrangements possible per grouping is mentioned, that is how many times can the objects be switched around or ordered so that the unordered sequence is still the same? In this case, we have identified two groups of distinct objects, so we cannot move them around or else they will represent a different grouping.

However, we can swap individual stars with other stars and that will be unnoticeable, counting as the same grouping. So for the bars, how many orderings are possible? Well, that is again just permutating the number of bars into the same number as slots. So we have $P(n-1, n-1)$, and by now, you might have noticed that any m permutating itself is $m!$, so we have $(n-1)!$ as our total possible orderings of bars that will leave the same appearance, as all bars are also indistinguishable from other bars. We do the same for stars, which is just again k stars that can be re-arranged $k!$ different ways.

Therefore, by the multiplication principle, the total possible arrangements per grouping is $k!\,(n-1)!$. Let us plug this back into our equation, along with our other discovery earlier that the total arrangements possible is $(k+(n-1))!$:

$$\#\text{ of unordered groupings} = \frac{\text{total arrangements possible}}{\text{arrangements possible per grouping}}$$

$$\#\text{ of unordered groupings} = \frac{(k+(n-1))!}{(n-1)!\,k!},$$

which is equal to

$$C(k+(n-1), k) = C(k+(n-1), n-1),$$

$$\boxed{\binom{k+(n-1)}{k} = \binom{k+(n-1)}{n-1}}.$$

Again, we successfully derived Stars and Bars (Definition 1.4.3).

Great! We have now explored three ways of obtaining this answer. We can now fill in our last box in our table.

	Ordered Sequence	Unordered Sequence
Repetition Allowed	n^k	$\binom{k+(n-1)}{k} = \binom{k+(n-1)}{n-1}$
Distinct Choices	$P(n,k)$	$C(n,k)$

Table 1.4.2: Choosing k from $\{1, 2, \ldots, n\}$

Let's now test our new skill with an example.

Example 1.4.4

You are baking cookies and have 6 tablespoons of sugar. You want to distribute this sugar among 4 different bowls to mix with various ingredients. The bowls can hold different amounts of sugar, including zero tablespoons. How many ways can you distribute the sugar among the bowls?

Solution. This is an application of stars and bars. We are trying to distribute 6 tablespoons of sugar into 4 different bowls, or as we call "bins." So $k = 6$ and $n = 4$. In order to separate this into 4 bowls, we need 3 bars. Each tablespoon of sugar can be represented by a star.

By the stars and bars method, the number of ways to distribute the tablespoons of sugar is (stars+bars) choose either stars or bars. The total of stars and bars is $6+3 = 9$. Let's use stars, and so we do 9 total positions choose 6 stars to find the total unordered groupings possible, resulting in

$$\begin{aligned}\binom{9}{6} &= \frac{9!}{(9-6)!\,6!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9\cdot 8\cdot 7\cdot 6!}{3!\,6!} \\ &= \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} \\ &= \frac{(3\cdot 3)\cdot(4\cdot 2)\cdot 7}{3\cdot 2\cdot 1} \\ &= \frac{3\cdot 4\cdot 7}{1} \\ &= \boxed{84}\end{aligned}$$

$\therefore$ There are 84 ways of distributing the 6 tablespoons of sugar among the 4 bowls.

†

Youtube Lectures 1.4

1. Stars and Bars (Part 1) - Counting Principles
2. Stars and Bars (Part 2) - Counting Principles

Exercises 1.4

Exercise 1.4.1

How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are neither subsets of $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$?

Exercise 1.4.2

The number of ordered quadruples (x_1, x_2, x_3, x_4) of odd positive integers such that $x_1+x_2+x_3+x_4 = 36$ can be expressed as $a \cdot b \cdot c$ such that a, b, c are prime numbers. Find the sum $a + b + c$.

Exercise 1.4.3

How many rearrangements of the letters of "SEEDSEED" do not contain the substring "SEED"? (For instance, one such arrangement is SEESEDED.)

Exercise 1.4.4

If 240 equals xyz, the product of the positive integers x, y and z, find the number of distinct ordered triples (x, y, z) such that x is a multiple of 2, y is a multiple of 3 and z is a multiple of 5.

Exercise 1.4.5

Let n be the number of subsets with cardinality ("size") of 5 that can be chosen from the set of the first 14 natural numbers so that at least two of the five numbers are consecutive. Find $n\%1000$ (the remainder when n is divided by 1000).

Exercise 1.4.6

There are 12 students in a class, and they need to be divided into five project teams. Three of the teams will consist of 2 students each, and the other two teams will have 3 students each. How many distinct ways can these teams be organized?

Exercise 1.4.7

There are 77 one-dollar bills to be distributed to 7 people so that no person receives less than $10. Compute the number of distinct ways in which this can be done.

Exercise 1.4.8

In a shooting competition, there are eight clay targets suspended in three columns: two columns with three targets each and one column with two targets. The shooter must break all the targets by following these rules:

1. The shooter selects a column from which a target is to be broken.
2. The shooter must then break the lowest remaining target in that chosen column.

Under these rules, how many different sequences of shots can the shooter make to break all the targets?

Exercise 1.4.9

Four partners, Alex, Ben, Carol, and Dana, are dividing 16 equal shares of a business among themselves. Here are the conditions they must follow:

1. Each partner must get a positive integer number of shares, and all 16 shares must be given out.
2. No partner can own more shares than the sum of the shares owned by the other three partners

Given these conditions, and assuming that shares are indistinguishable, but people are distinguishable, how many different ways can the shares be allocated to the partners?

Exercise 1.4.10

Consider the set $\{1, 2, 3, \ldots, 15\}$. Determine the number of non-empty subsets T that satisfy the following conditions:

1. The subset T does not include any pair of consecutive numbers.
2. If T has k elements, then it contains no numbers smaller than k.

1.5 The Principle of Nothingness

Let me quickly introduce a simple yet briefly important concept to mention before we segway into important combinatorial identities and techniques. These include important distinctions that will appear in future chapters.

The empty set, $\{\}$ or $\emptyset$, is a set that contains no elements. What that means is that the size of the empty set, known as the cardinality of a set, is equal to zero. That is, $|\emptyset|= 0$. We will later mention this as it is defined by the Principle of Nothingness.

Definition 1.5.1 (The Principle of Nothingness Part 1)
The sum of items n in the empty set is equal to zero:

$$\sum_{n\in\emptyset} a_n = 0.$$

This definition defines that even if there are no objects in a set, then the total count is zero. That's a pretty intuitive definition.

Example 1.5.2
How many ways are there to tile a 1×1 board with 2×2 pieces?

Solution. 0 ways, this is impossible. Easy, right? Let's try another one. †

Example 1.5.3
How many ways are there to tile a $n \times n$ board with 1×1 or 1×2 pieces if $n = 0$?

Solution. There is only 1 way to do this. We can do this with no pieces. As soon as the question was asked, this task was done and no task has to be performed. In other words, there is only 1 way to do nothing. †

Example 1.5.4
Find the value of $|\{\emptyset\}|$.

Solution. Either surprisingly or not surprisingly, the answer is 1. Why is the cardinality of this set containing an empty set equal to one? Well, the empty set is something and that is considered an element of the set. That makes sense!

Let A, B be two sets with no elements. By definition, there are no elements in set A that are not in set B and vice versa. Therefore, this confirms that these two sets are equal. That is $A = B$, meaning that any set with no elements is the empty set, which proves its existence and uniqueness. †

Definition 1.5.5 (The Principle of Nothingness Part 2)
The empty set is unique. The Principle of Nothingness defines the existence and uniqueness of the empty set as axiomatic. This principle also defines other properties of the empty set ($\emptyset$):

1. The empty set is a subset for any set S: $\emptyset \subset S$.
2. The cardinality of an empty set is 0: $|\emptyset| = 0$.
3. The empty set is an identity element for intersection. For any set S, $S \cap \emptyset = \emptyset$.
4. The empty set is an identity for union. For any set S, $S \cup \emptyset = S$.

Condition (1) is defined due to the empty set not violating any other condition of being a subset of a set, as it contains no elements that are not elements of the set S.

Note that choosing any $k = 1$ items from a set of n distinct items despite the selection being ordered or unordered, or if the items chosen have to be distinct or not, the result is always equal to 1. This is obtained by the formulas in each box of Table 1.4.2 after we set $k = 0$. But why is this true? Well, it is again the Principle of Nothingness. We are asked to find how many ways to choose 0 objects from a set, and that has already been done. Again, there is one way of doing nothing since the empty set is unique.

Despite the rudimentary feeling of this property, you will find several useful applications of this principle in certain situations, such as those where we deal with Linear Recurrences.

Youtube Lectures 1.5

1. The Principle of Nothingness - Counting Principles

Exercises 1.5

Exercise 1.5.1
Given a set $A = \{a, b, c\}$, how many subsets of A have exactly zero elements? Use the Principle of Nothingness to justify your answer.

Exercise 1.5.2
Let $B = \{1, 2, 3\}$ and $C = \emptyset$. What is the intersection $B \cap C$, and why?

Exercise 1.5.3
In how many ways can you tile a 2×0 board with dominoes (pieces of size 1×2)? Explain your reasoning using the Principle of Nothingness.

Exercise 1.5.4

How many ways can you choose zero items from the set $D = \{w, x, y, z\}$? Explain why this is consistent with the Principle of Nothingness.

CHAPTER 2

Combinatorial Identities and Techniques

2.1 Pascal's Triangle

You may have heard this somewhere when dealing with binomial expansion in your pre-calculus class. Let's start with the example $(x+y)^0$, which we will obtain 1, which has a coefficient of 1. Next, if you take the polynomial $(x+y)^1$, you will obtain the expanded form $x+y$. These terms have coefficients 1, 1. If we take $(x+y)^2$, then we will obtain $x^2+2xy+y^2$. These terms have coefficients 1, 2, 1. What about $(x+y)^3$? That is $x^3+3x^2y+3xy^2+y^3$, with coefficients 1, 3, 3, 1.

The next $(x+y)^n$ where $n=5$ would have coefficients 1, 4, 6, 4, 1. Do you see the pattern?

Firstly, both the right and left outer terms are always 1. Secondly, notice how each term is formed by the sum of the two terms directly above it in the previous row if each n was placed on a row and centralized like shown below, forming a triangle of row n:

$$\begin{array}{r|ccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Figure 2.1.1: Pascal's Triangle (5 rows)

This is called Pascal's triangle, named after Blaise Pascal, a 17th-century French mathematician who was probably the first person to derive this identity. This pattern indeed follows for every $n > 4$.

Definition 2.1.1 (Pascal's Triangle)
Pascal's Triangle is a triangular array of numbers where each row represents the polynomial expansion of $(x + y)^n$. Each term in each following row is equal to the sum of the two terms directly above it.

A third pattern emerges when you add up all the numbers in each row. In our previous example, we will obtain 1, 2, 4, 8, and 16. This is 2^n, or the number of total subsets that can be formed from an n-set (a set of n elements). But notice that the number of subsets of size k that can be formed from an n-set is $\binom{n}{k}$.

Remember in Chapter 1 when we explored the exhaustive sum to obtain the total sum of 2^n? For $n = 0$, only a subset of size 0 is possible, which is $\binom{0}{0}$. For $n = 1$, sizes 0 and 1 are possible, and we had $\binom{1}{0} + \binom{1}{1} = 2$. The more we go, the more we realize that $n + 1$ sized subsets are available to be summed up to find the total number of subsets that can be formed for each n-set.

Now, look at each row of Pascal's triangle. For the row $n = 0$, there is one term. For $n = 1$, there are two terms. For $n = 2$, there are 3 terms, and so on. This agrees with the number of k-sized subsets in the sum to create the 2^n total.

The next question in mind would be if the number of k-sized subsets, which is counted by $\binom{n}{k}$, matches the kth term in Pascal's triangle starting with $k = 0$ as the leftmost term in the row n. Surprisingly, we find that this indeed is true.

Take two examples: (1) the third term on the fifth row (when $k = 2$ and $n = 4$) and (2) the second term on the fourth row (when $k = 1$ and $n = 3$). Examples (1) and (2) are 6 and 3, respectively, on Pascal's Triangle as shown in Figure 2.1.1. Let's test this with our formula for counting subsets, $\binom{n}{k}$.

Example (1) should be

$$\begin{aligned}\binom{4}{2} &= \frac{4!}{(4-2)!\,2!} \\ &= \frac{4!}{2!\,2!} \\ &= \frac{4 \times 3 \times 2 \times 1}{4} \\ &= \boxed{6},\end{aligned}$$

and Example (2) should be

$$\begin{aligned}\binom{3}{1} &= \frac{3!}{(3-1)!\,1!} \\ &= \frac{3!}{2!}\end{aligned}$$

$$= \frac{3 \times 2 \times 1}{2 \times 1}$$
$$= \boxed{3}.$$

It turns out that this expression, $\binom{n}{k}$, accounts for every single item in Pascal's Triangle and can be used to determine the coefficient on each term for any binomial expansion. This is called the **binomial coefficient**.

Let's recreate Figure 2.1.1 with our new discovery, replacing each term with $\binom{n}{k}$:

$$\begin{array}{r|ccccccccc}
n=0 & & & & & \binom{0}{0} & & & & \\
n=1 & & & & \binom{1}{0} & & \binom{1}{1} & & & \\
n=2 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
n=3 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
n=4 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Figure 2.1.2: Pascal's Triangle (5 rows) with Binomial Coefficients

We understand the main property of Pascal's triangle works for every term, including the outer ones by assuming the invisible terms (terms where k is larger than n or less than 0) to be equal to 0. So what if we took out an arbitrary term to represent this property as an identity for binomial coefficients?

$$\binom{n}{k} \qquad \binom{n}{k+1}$$
$$\binom{n+1}{k+1}$$

By the property of Pascal's Triangle, we obtain Pascal's Identity:

Theorem 2.1.2 (Pascal's Identity)

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1},$$
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Another feature of Pascal's Triangle that may be quite obvious is its symmetry. That is, listing the numbers from the left to right is equal to the list of numbers going from right to left. In binomial coefficient terms, this is:

Theorem 2.1.3 (Pascal's Symmetry)

$$\boxed{\binom{n}{k} = \binom{n}{n-k}}.$$

We have already explored this feature back in Chapter 1 when we discovered this binomial coefficient formula to be equal to $C(n, k)$. Of course, we could simply solve this algebraically by plugging in each binomial coefficient expression for the formula from Definition 1.3.6:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

I encourage you to try this, but let's prove it in a more combinatorial way. This will be a simple proof, as follows:

Proof. We would like to prove that

$$\binom{n}{k} = \binom{n}{n-k}.$$

The left-hand side (LHS) in Theorem 2.1.3 is equal to , or choosing k objects from n objects. This leaves $n - k$ items left. Similarly, the right-hand side (RHS) is the same as choosing $n - k$ objects from a set of n objects. This leaves k items left.

In other words, say you have a pile of n objects. On the LHS of Theorem 2.1.3, we have $\binom{n}{k}$, meaning we are choosing k objects that form a new pile, leaving $n - k$ objects left in the original pile. On the RHS, we have $\binom{n}{n-k}$. We again start with n objects, and that RHS expression means we are choosing $n - k$ objects into a new pile, leaving k objects in the original pile.

In both instances, we form two groups of k and $n - k$ objects, which are symmetrical in the perspective of combinations, and support unordered selections. $\square$

There are a few more noticeable features of Pascal's Triangle that will be discussed in the next topic about the Binomial Theorem. Feel free to look for these on your own.

Youtube Lectures 2.1

1. Introduction to Pascal's Triangle

Exercises 2.1

Exercise 2.1.1

Construct an alternative proof of Theorem 2.1.3 with a set-subset argument.

Exercise 2.1.2

Conjecture the sum of each diagonal of Pascal's Triangle (8 diagonals are given in their own distinct color, with the yellow representing incomplete diagonals. You may keep going and add more rows and make new diagonals if needed):

$$\begin{array}{llllllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}$$

Then, prove your conjecture.

Exercise 2.1.3

Given $(x+y)^{11}$, use Pascal's Triangle to find the coefficient of x^5y^6. Then, find the sum of all coefficients of that row.

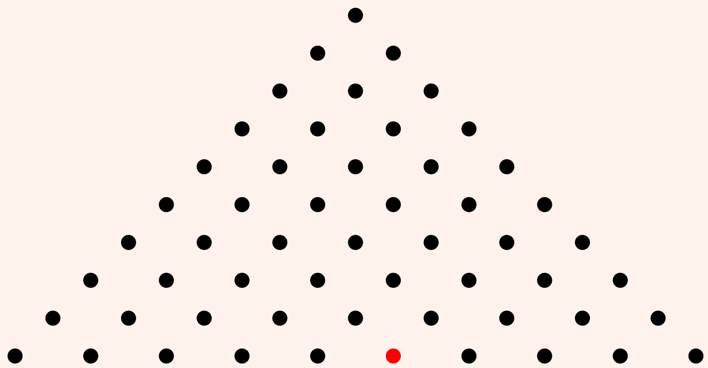
Exercise 2.1.4

In Pascal's Triangle, each entry is the sum of the two entries above it. In which row of Pascal's Triangle do three consecutive entries occur that are in the ratio $3:4:5$? (AIME)

Exercise 2.1.5

A Galton Board is a flat surface that is placed upright as shown below. A ball is to be dropped on top of the top pin. It has a 50% chance of going left or right and that happens for every pin it encounters. If we assume a completely statistical simulation and 1 ball is dropped on the top pin, what are the chances of the ball encountering the red pin (the 6th pin on the 10th row going from left to right)?

Hint: In how many possible paths does the ball encounter each pin in the 2nd row? 3rd? 4th? 10th?



2.2 Binomial Theorem

We have explored the coefficient of each term for an expansion given as the binomial coefficient, but what are the other parts of the term? Namely, the x and y terms of the expanded polynomial when we expand some $(x+y)^n$. Let's first think of the binomial coefficient as the number of times the other parts of the term show up, that is the sum of the times $y^a x^b$ terms show up. We know that in the expansion, $(x+y)$ shows up n times, meaning that each term in the simplified expansion has some unique combination of x's and y's. Here is a better visual:

$$(x+y)(x+y)(x+y)(x+y)\dots(x+y) \text{ (n times)}$$

One possible initial expansion term would be by multiplying either x or y terms from each of the n binomials as shown:

$$(x+\mathbf{y})(x+\mathbf{y})(x+\mathbf{y})(x+\mathbf{y})\dots(\mathbf{x}+y) \text{ (n times)}$$

Say that there were 8 bolded y and $n-8$ bolded x (notice that every term must have a total of n x's and y's, since we must multiply one term from every binomial from the pool of n binomials). Then, that term would be expressed by

$$Ay^8x^{n-8},$$

where A is the coefficient of that term.

Next, notice that the maximum power of x is n, consisting of the case where every term multiplied is the x term and y would be multiplied no times. The lowest power possible power for x is 0, where it is multiplied zero times and y is multiplied all n times. These bounds are also the same for y.

Also notice that in either of these two most extreme cases, there is only one way to have each of them happen, as we are viewing unordered selection so a selection of all y or all x can only happen once. This is also supported by the binomial coefficients from the Pascal's Triangle.

Since we now understand that the powers of y and x must add up to n, we know that each term must be in some form $Ax^k(y^n - k)$, where A is the coefficient. But what is equal to the coefficient?

Well, we know that this counts the number of times $x^k y^{n-k}$ shows up in the expanded form. Additionally, we know there are a total of n binomials, and from each binomial, either x or y are chosen.

So, if we wanted to know how many ways from the n binomials to pick exactly k x's and $n - k$ y's, then we can just choose all the k x's and the remaining spots left will automatically need to be y's. This is just $\binom{n}{k}$. We could have also chosen $n - k$ y terms, obtaining us $\binom{n}{n-k}$, leaving the exact remaining unpicked k spots to be x.

These two ways are the same and once again show the symmetry of Pascal's triangle. This coefficient is again the binomial coefficient, as we have brought up in the previous chapter. Therefore,

$$A = \binom{n}{k} = \binom{n}{n-k}$$

(the second expression of n choose k is more often used than the $n - k$ one due to simplicity).

Theorem 2.2.1 (The Binomial Theorem)
The expansion of any binomial is given by the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Notice that

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

due to both the symmetry of the binomial coefficients in Pascal's Triangle and the commutative property of addition, where we know

$$(x+y)^n = (y+x)^n.$$

A very important yet simple technique for proving a theorem or conjecture true is a **proof by mathematical induction**. This has a large range of uses in number theory, algebra, graph theory, geometry, computer science, and many more topics. For combinatorics, too, this is a widely used tool to confirm many combinatorial identities, specifically dealing with binomial coefficients, as you will see right now and in the exercises given. Proof by induction involves two main components:

1. Base Case
2. $n+1$ or The "Inductive" Step

Basically, if we know the base case is true, and we are able to prove the case is true for all $n+1$, then any n above the base case would have to be true. This is the essence of the proof. More interesting examples (such as proving the sum of first n integers as $\frac{n(n+1)}{2}$) of this will be used in Chapter 5 dealing with recurrence relations.

Let's jump right in and use mathematical induction to prove the binomial theorem.

Proof. Base case:

$$n = 0 : (x+y)^0 = 1.$$

The binomial theorem gives

$$\begin{aligned}
(x+y)^0 &= \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k} \\
&= \binom{0}{0} x^0 y^0 - 0 \\
&= \binom{0}{0} \\
&= \frac{0!}{(0-0)!\,0!} \\
&= \frac{0!}{(0-0)!\,0!} \\
&= \boxed{1}.
\end{aligned}$$

Inductive step:

$$\begin{aligned}
(x+y)^{n+1} = (x+y)(x+y)^n &= (x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= x\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \text{(Distributive Property)} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}
\end{aligned}$$

Now let's use an index shift, setting $m = k+1$, or $k = m-1$, for the first sum in order to obtain the same x, y powers. Keep in mind when the lower bound index is shifted, the upper bound must also be changed to have the same difference between the bounds, in this case, n becomes $n+1$:

$$\sum_{m-1=0}^{n+1} \binom{n}{m-1} x^m y^{n-(m-1)} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}$$
$$= \sum_{m=1}^{n+1} \binom{n}{m-1} x^m y^{n-m+1} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}$$

Now let's try to make the two sums into one sum by rewriting the second expression's lower bound starting at $k = 1$ by "spitting" out the term in which $k = 0$, adding it to the side.

$$\sum_{m=1}^{n+1} \binom{n}{m-1} x^m y^{n-m+1} + \left(\sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + \binom{n}{0} x^0 y^{n-0+1} \right)$$
$$= \sum_{m=1}^{n+1} \binom{n}{m-1} x^m y^{n-m+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + \frac{n!}{(n-0)!\,0!} y^{n+1}$$
$$= \sum_{m=1}^{n+1} \binom{n}{m-1} x^m y^{n-m+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1}$$

We need to also fix the $n + 1$ as the last term of the left sum in order to combine the two sum expressions into one sum (as we need the same bounds). So let's take out the term in which $m = n + 1$ from the first sum.

$$\left(\sum_{m=1}^{n} \binom{n}{m-1} x^m y^{n-m+1} + \binom{n}{n+1-1} x^{n+1} y^{n-(n+1)+1} \right) + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1}$$
$$= \sum_{m=1}^{n} \binom{n}{m-1} x^m y^{n-m+1} + \binom{n}{n} x^{n+1} y^{n-n-1+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1}$$
$$= \sum_{m=1}^{n} \binom{n}{m-1} x^m y^{n-m+1} + \frac{n!}{(n-n)!\,n!} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1}$$
$$= \sum_{m=1}^{n} \binom{n}{m-1} x^m y^{n-m+1} + x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + y^{n+1}$$
$$= \sum_{m=1}^{n} \binom{n}{m-1} x^m y^{n-m+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n-k+1} + x^{n+1} + y^{n+1}$$

These two sums now have the same limits and can be combined. Note that m and k in this case behave the exact same, so we can express m as k as well, all under one sum.

$$\sum_{k=1}^{n} \left(\binom{n}{k-1} x^k y^{n-k+1} + \binom{n}{k} x^k y^{n-k+1} \right) + x^{n+1} + y^{n+1}$$
$$= \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n-k+1} + x^{n+1} + y^{n+1}$$

Now, we can use Theorem 2.1.2 and substitute n with $n+1$ (note this substituted expression happens to work since we are now dealing with $n+1$ instead of n):

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ (Theorem 2.1.2)}$$
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k},$$
$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

So let's plug in our new expression in the summand:

$$\sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n-k+1} + x^{n+1} + y^{n+1}$$

We are almost done. Now, let's reverse-engineer the two non-summation expressions back into the summation. The x^{n+1} term can be thought of as the term when $k = n+1$, where we would then obtain

$$\binom{n+1}{n+1} x^{n+1} y^{n-n-1+1} = x^{n+1}$$

(any number p choose itself is equal to 1, and $y^0 = 1$). This extra term that we account for essentially just stretches the summation's upper bound from $k = n$ to $k = n+1$.

$$\sum_{k=1}^{n+1} \binom{n+1}{k} x^k y^{n-k+1} + y^{n+1}$$

Let's try to do the same thing to get the y^{n+1} term inside the summation as well. That is the case when the summation has $k = 0$, as we will get

$$\binom{n+1}{0} x^0 y^{n-0+1} = y^{n+1}$$

(since any number q choose 0 is equal to 1 and x^0 is again equal to 1). This extra term essentially stretches the summation's lower bound from $k = 1$ to $k = 0$. So, we finally obtain:

$$\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n-k+1},$$
$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

By the Binomial Theorem,

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

We have proved the inductive step because we have shown that by assuming the inducted Binomial Theorem formula to be true for any n and plugging into the expression equal to $(x+y)^{n+1}$, the formula also works for the $n+1$ case. Since we know the base case is true for the formula (where $n=0$), we know the Binomial Theorem also works for the next case ($n=1$), meaning it also works for the next one, and so on. This proves that the Binomial Theorem works for every $n \geq 0$.

$$\therefore \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}.$$

□

The Binomial Theorem is very useful in uncovering some cool identities dealing with Pascal's Triangle. One important example is the addition of entire rows, which we had previously evaluated as the same as the total number of subsets of any size that can be formed from a set of n elements, or 2^n. The Binomial Theorem provides us a very easy alternative to achieve this result.

Since the Binomial Theorem has the binomial coefficient $\binom{n}{k}$ in its summation, we can utilize this feature to find the summation of all binomial coefficients in a row to find the sum by making both x and y powers to always become 1. This will cancel out the variables and leave only the binomial coefficients in the sum.

Proof. Let $x=1$ and $y=1$, exploiting the identity of 1 as 1 raised to any power is equal to 1:

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (x)^k (y)^{n-k} \text{ (Theorem 2.2.1)} \\
&= \sum_{k=0}^{n} \binom{n}{k} (1)^k (1)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}$$

This was extracted from the RHS of the Binomial Theorem (Theorem 2.2.1), which is equal to the LHS:

$$\begin{aligned}
(x+y)^n &= (1+1)^n \\
&= 2^n
\end{aligned}$$

Therefore, we obtain that the total subsets of any size able to be formed from a n-set or the sum of a row n on Pascal's Triangle is equal to 2^n:

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} = 2^n}$$

□

Another very interesting feature that you may or may not have noticed can also be extracted from Pascal's Triangle. This is the alternation identity of Pascal's Triangle, in which the sum of all even k numbers are equal to the sum of all odd k numbers in any row except row 0.

Example 2.2.2

How can we find the alternation of only the binomial coefficients using the Binomial Theorem?

Solution. Well, we know alternation can be caused if either all the even or odd numbers are negative, while the other is positive. Then, we would be able to find their sum and it should be equal to 0.

If we let $x = -1$ and $y = -1$, this would not work because we would have $(-1)^k(-1)^{n-k}$ which results in sums of all negative binomial coefficients if n is odd and all positive if n is even, given by $(-2)^n$.

So, suppose we only want one alternation instead of a double alternation. In that case, we can set one of the variables to be 1 as a placeholder and the other as -1 for only one alternation, as $(-1)^{\text{odd}}$ would yield negative terms and if it is $(-1)^{\text{even}}$, then it would yield positive terms.

So let's let $x = -1, y = 1$:

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k} \\
&= \sum_{k \in \mathbb{Z}^{\text{even}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} (-1)^k + \sum_{k \in \mathbb{Z}^{\text{odd}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} (-1)^k \\
&= \sum_{k \in \mathbb{Z}^{\text{even}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} (1) + \sum_{k \in \mathbb{Z}^{\text{odd}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} (-1) \\
&= \sum_{k \in \mathbb{Z}^{\text{even}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} - \sum_{k \in \mathbb{Z}^{\text{odd}}, k \in \{1,2,\ldots,n\}} \binom{n}{k}
\end{aligned}$$

We know that this is equal to $(x + y)^n$, and we can plug in $x = -1, y = 1$:

$$\sum_{k \in \mathbb{Z}^{\text{even}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} - \sum_{k \in \mathbb{Z}^{\text{odd}}, k \in \{1,2,\ldots,n\}} \binom{n}{k} = (-1+1)^n,$$

$$(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots) - (\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots) = (0)^n,$$

$$(\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots) - (\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots) = 0$$

$$\therefore \boxed{\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots}$$

†

There are more amazing Pascal Triangle features and identities that you can discover through the Binomial Theorem. Some more will be included in Exercises 2.2.

But just before we move onto the exercises, we will go over a cool application of the binomial coefficient and combinatorial thinking for our transition to the next section. This is a common grid walking problem—a problem that counts the number of paths you can walk across a grid subject to certain restrictions.

Example 2.2.3

James wants to get from his school back to his house. How many ways can James walk across a $k \times n$ grid in a path in which he may only go up or right per move to reach his destination? The grid is given below.

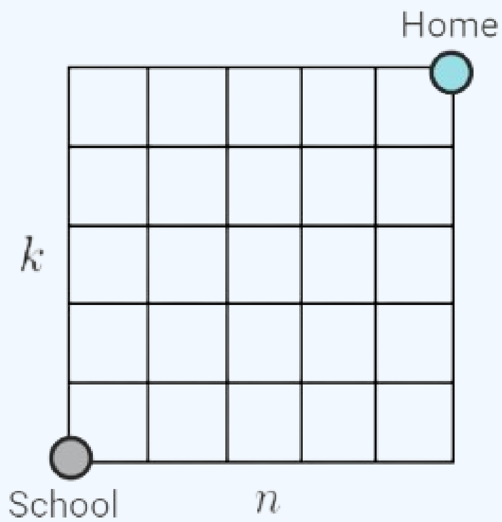


Figure 2.2.1: James' $k \times n$ Grid Walk

Solution. This problem is actually pretty simple. The main part of the problem is realizing the inevitable pattern across every successful path from the school to James' home. That is, every successful path that he takes involves exactly n steps right and k steps up. This is because he cannot go backwards as he only has the choices of going up or to the right.

For example, this example path from the school to James' house takes exactly 5 steps right and 5 steps up (assuming that $n = 5, k = 5$ for the sake of this example, although n, k are arbitrary variables and do not have to be the same):

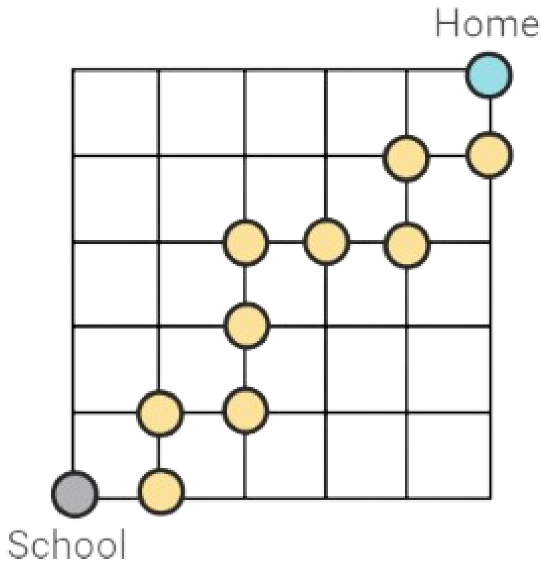


Figure 2.2.2: Example Grid Walk

Therefore, in a successful path, James takes n steps right and k steps up, making that $n+k$ total steps that he will take to get home. Then this becomes more apparent that we are just determining how many ways the total number of steps can be distributed into a sequence of length $n+k$ with n spots labeled as right moves and k spots labeled as upward steps.

Since choosing the spots for either of the two distinct labels will automatically leave the rest of the spots for the other label, this is just

$$\boxed{\binom{n+k}{k} = \binom{n+k}{n}}.$$

†

By the way, this is supported by the multiplication principle as there is only one way to choose each spot for the rest of the spots, making the number of ways to choose the first label times 1 multiplied by itself however many spots left, which is just the same as the number of ways to choose the first label.

Definition 2.2.4 (Total Paths of a Grid Walk)
The total paths to walk in a grid in which you may only go north or east in a $n \times k$ grid is given by

$$\binom{n+k}{k} = \binom{n+k}{n}.$$

Interesting, right?

Youtube Lectures 2.2

1. Deriving the Binomial Theorem: A Proof by Induction

2. Proving Pascal's Identity: An Algebraic Method

Exercises 2.2

Exercise 2.2.1
Find the identity that appears for the binomial expansion of $(1+x)^n$ when $x = 2$.

Exercise 2.2.2
Find the identity that appears for the binomial expansion of $(1+x)^n$ when $x = -2$.

Exercise 2.2.3
In the expansion $(x - 4z^2)^5$, find the coefficient of x^2z^6.

Exercise 2.2.4
Find the center term of $(4x - 3y)^6$.

Exercise 2.2.5
Prove the following by induction:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Exercise 2.2.6
Use proof by induction to prove

$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1.$$

2.3 Double Counting

Remember the summation of a Pascal's Triangle row formula where 2^n was equal to $\sum_{k=0}^{n}\binom{n}{k}$?

Well when we interpreted this was the equivalent to the summation of all sized subsets using our knowledge that $\binom{n}{k}$ is the amount of subsets that can be formed of size k from a n-set, we actually used the method of double counting to evaluate that the sum must be the same as the 2^n since that is the total number of subsets that may be formed.

Definition 2.3.1 (Double Counting)

double counting, as the name suggests, is using two scenarios or different representations of notation given in the LHS and RHS of a combinatorial equation. Although a bit less formal than proof by induction, double counting is a useful alternative method that is useful in developing combinatorial thinking and argument for proving conjectures, theorems, and identities.

A diagram representation of this technique is shown on the next page.

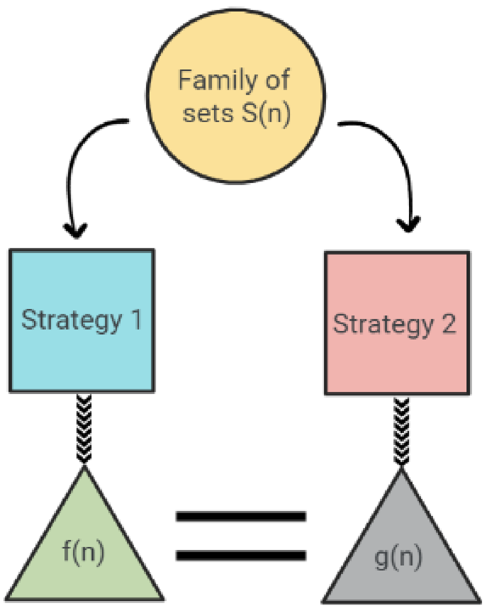


Figure 2.3.1: Double Counting Diagram

Let's apply this technique to prove the canonical Pascal's Identity that we derived earlier from Pascal's Triangle (Theorem 2.1.2).

Proof. We would like to prove the identity using Double Counting:

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k} \text{ (Theorem 2.1.2)}$$

Let's start by interpreting the RHS. The RHS counts the number of ways to choose a subset of size k from $\{1, 2, \ldots, n\} \to \binom{n}{k}$. Now let's think of this idea represented by the RHS in terms of the two binomial coefficients on the LHS. We are essentially trying to split this case into two cases.

We also know that the only difference between $\binom{n-1}{k}$ and $\binom{n}{k}$ is choosing subsets that contain the number n.

Therefore, we can just split this into two cases where one case accounts for choosing subsets of size k that do not contain n, which we know is $\binom{n-1}{k}$, and subsets that contains n. Splitting the LHS into two cases, case 1 and case 2, we get the following:

Case 1 (subsets not containing n): Counts the number of subsets of size k that do not contain n from $\{1, 2, \ldots, n\}$, which is just the number of subsets of size k that can be chosen from $\{1, 2, \ldots, n-1\}$. This is clearly $\binom{n-1}{k}$.

Case 2 (subsets containing n): Counts the total number of subsets of size k from the set $\{1, 2, \ldots, n\}$ that contain n. This means n must be an element, meaning that we are actually choosing $k-1$ elements as one of the k elements we are looking to choose has already been chosen. We also must bear in mind we are now choosing from the set $\{1, 2, \ldots, n-1\}$ since we cannot choose n again, as problems dealing with combinations have distinct choices/selection, and in this problem of k elements from the set $\{1, 2, \ldots, n\}$. Therefore, we are choosing subsets with $k-1$ elements from $\{1, 2, \ldots, n-1\}$, which is just $\binom{n-1}{k-1}$. This matches the first binomial coefficient in the LHS.

So, the RHS counts the number of subsets of size k from $\{1, 2, \ldots, n\}$ and the LHS counts subsets of size k from $\{1, 2, \ldots, n-1\}$ which do not contain n and also counts the number of subsets of size $k-1$ from $\{1, 2, \ldots, n-1\}$ that do contain n. Therefore, by the rule of sum/addition principle, we have counted the same number in two different ways and so we have proved that

$$\boxed{\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}}.$$

$\square$

Of course, we can always solve this using either proof by induction or just algebra. Let me show you the algebra this real quick, just to confirm our answer even more. All we have to do is use the binomial coefficient formula (from Definition 1.3.2) and plug in for every term. The binomial coefficient formula is shown below.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Proof. Goal: Prove Pascal's Identity using an algebraic approach.

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} \text{ (Theorem 2.1.2).}$$

The first binomial coefficient would yield

$$\begin{aligned}\binom{n-1}{k-1} &= \frac{(n-1)!}{(n-1-(k-1))!\,(k-1)!}\\ &= \frac{(n-1)!}{(n-1-k+1)!\,(k-1)!}\\ &= \frac{(n-1)!}{(n-k)!\,(k-1)!}\end{aligned}$$

The second binomial would be equal to

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!\,k!}.$$

Now we substitute each binomial coefficient for these simplified forms into the original equation (Theorem 2.1.2):

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$

$$\frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} = \frac{n!}{(n-k)!\,k!}.$$

Next, multiply both sides by $k!$:

$$\frac{(n-1)!\,k!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!} = \frac{n!}{(n-k)!}.$$

Divide by $(n-1)!$ on both sides:

$$\frac{k!}{(n-k)!\,(k-1)!} + \frac{1}{(n-1-k)!} = \frac{n!}{(n-k)!\,(n-1)!}.$$

Multiply both sides by $(n-k)!$:

$$\frac{k!}{(k-1)!} + \frac{(n-k)!}{(n-1-k)!} = \frac{n!}{(n-1)!}.$$

Expand numerator factorials:

$$\frac{k\times(k-1)!}{(k-1)!} + \frac{(n-k)\times(n-k-1)!}{(n-k-1)!} = \frac{n\times(n-1)!}{(n-1)!}.$$

Finally, simplify and we get:

$$\begin{gathered}k+(n-k)=n,\\ k+n-k=n,\\ \boxed{n=n}\ \checkmark\end{gathered}$$

$$\therefore \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

$\square$

I encourage you to try and prove this through induction with the theorem being your inductive hypothesis. For now, though, we will get back to double counting.

I would like to mention a stylistic alternative for double counting. We applied the use of subsets/sets to represent the binomial coefficients. But another common approach can be used similar to how we used the subsets/sets representation, something called committee forming. This is up to your choice to use this method or the subsets/sets method to think of these combinatorial problems.

Definition 2.3.2 (Committee Forming)

In the committee forming approach, binomial coefficients are represented by committee choosing.

Let n and k be non-negative integers with $0 \leq k \leq n$. The binomial coefficient $\binom{n}{k}$—which represents the number of ways to choose k elements from a set of n distinct elements—can be interpreted as the number of ways to form a committee of k members from a group of n people.

That is,

$$\binom{n}{k} = \text{the number of distinct k-member committees that can be formed from a set of n individuals.}$$

This is similar to choosing subsets of size k from a set of n distinct numbers. Since this is by choice/your preference, you may even create your own method. If needed, you could use other terms like "sub-committees," "board members," or "committee groups". Have a look at an example.

Example 2.3.3

Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Solution. Let's begin by assembling our thoughts into the RHS. Bluntly speaking, that is the amount of committees of size n that can be formed or chosen from a group of $2n$ people.

The LHS, on the other hand, is a bit more complicated. Let's start from one of the $\binom{n}{k}$ being multiplied to another $\binom{n}{k}$. We know straight away that that is choosing k people to form a committee from a population of n people.

But let's keep in mind what the RHS is doing, which is choosing from a population of $2n$. So we want to somehow make the LHS a split of the population $2n$ in order to represent it fully. We have two binomial coefficients:

$$\binom{n}{k}\binom{n}{k}.$$

Aha! Each of these are picking from a population of n people. Therefore, we are essentially splitting the $2n$ into two groups of n. Let's call these groups A and B. So then the first binomial coefficient can represent choosing k from group A. In order to form a group of n people in total, as requested by the RHS, we need to choose $n-k$ more for group B. That would be equal to $\binom{n}{n-k}$. But this isn't the second binomial coefficient that we got... Or is it?

Remember the symmetry of Pascal's Triangle? Indeed, from Theorem 2.1.3, $\binom{n}{n-k} = \binom{n}{k}$. Therefore, we obtain $\binom{n}{k}$ for choosing k people from group A and $\binom{n}{k}$ for choosing the rest from group B in order to form the committee of n people.

By the multiplication principle, we can choose k people from group A and choose $n-k$ people from group B in $\binom{n}{k}^2$ ways. But what about the summation from $k=0$ to $k=n$?

Well let's see. If the summand gives the number of ways to choose a total of n people by choosing a group of k people and another group of $n-k$ people, then... You got it (hopefully)! The summation accounts for every paired size of groups A and B.

For example, when $k=0$, we are counting how many ways to choose 0 members from group A and how many ways to choose $n-0=n$ members from group B. Therefore, we require the summand since we need to account for every single possibile distribution of sizes between the groups, which would be n possible unique distributions where k ranges from 0 to n.

Finally, we are done. Given a population size of $2n$, the LHS adds up all the possible ways to choose n people by choosing the possible ways to form two committees whose sum of members add up to n. That is, k people to put in "committee A" from group A and $n-k$ people to put in "committee B" from group B. The RHS counts this in a different yet equivalent way, by counting the ways to directly choose n people from the population. And we are done! †

Of course, you can also prove this using subsets in an identical way. However, there is a cool alternative you may consider, specifically when encountering the binomial coefficient of $\binom{2n}{n}$.

Instead of a word interpretation, this way will be using a drawing, more specifically a grid to help with the representation of the problem (for example, breaking one side into pieces that support counting the other side as a whole).

Proof. Suppose we have the following equation:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Recall Definition 2.2.4, where counting the total paths in which you may only travel in a direction towards the destination from one corner to another opposite corner in a $k \times n$ grid was proven to be

$$\binom{k+n}{n} = \binom{k+n}{k}.$$

Notice that when $k = n$, we obtain the total number of paths traveling from one corner to another of a $n \times n$ grid as $\binom{2n}{n}$, the RHS of the equation. Hence, let us break down that graph to represent the LHS. First, let's draw a $n \times n$ grid, which is shown below.

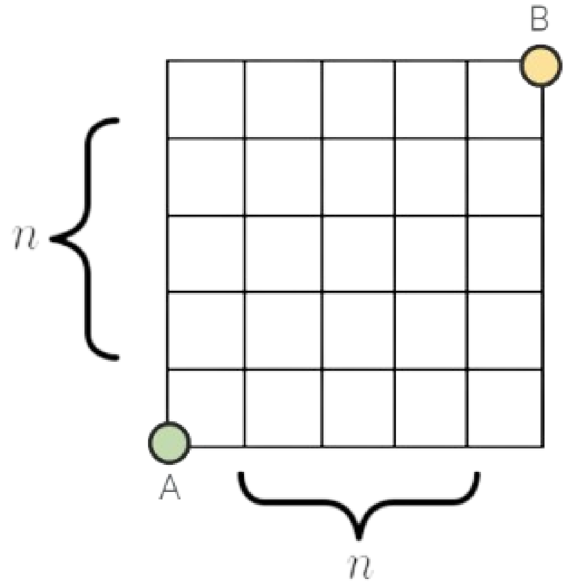


Figure 2.3.2: $n \times n$ Grid Walk

How can we represent the LHS, namely each of the $\binom{n}{k}$'s? Why not represent this as counting the number of possible paths of another, smaller grid. We can think of this using $n = x + k$ to represent the total number of paths that you can take traveling from one corner to the other in a $x \times k$ grid:

$$\binom{n}{k} = \binom{x+k}{k}.$$

This grid may be constructed by starting one corner at the starting point, A, traveling to the other corner labeled as C, as shown on the next page.

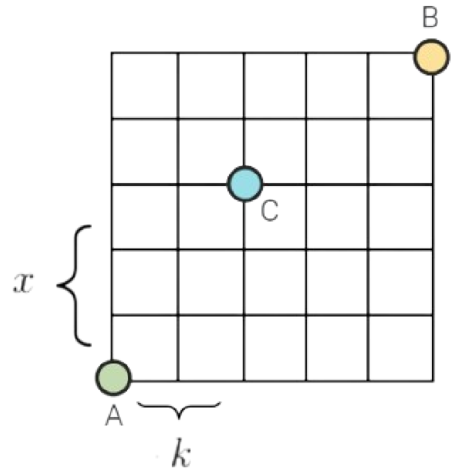


Figure 2.3.3: $n \times n$ Grid Walk with Point C

Great, but now, how do we account for the other binomial coefficient $\binom{n}{k}$? Well, we can think of it a different way, how do we achieve our goal of walking a path of $n+k$ in length from point A to point B?

Oh! We just have to find how many ways we can get from point C to point B. Let's label the dimensions of the rectangular grid with opposite corners C and B as $(n-k) \times (n-x)$, since we know the original grid has dimensions $n \times n$:

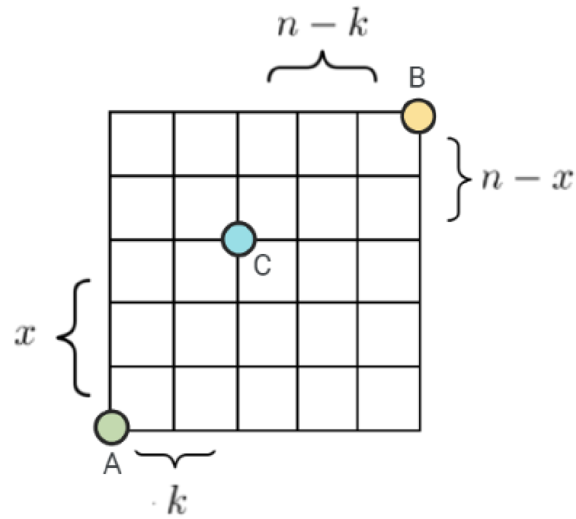


Figure 2.3.4: Newly Labeled $n \times n$ Grid Walk

We don't want to include even more variables to express the second binomial coefficient. We must realize that when we set $n = x + k$, that $n - x = k$. So, let's make that swap:

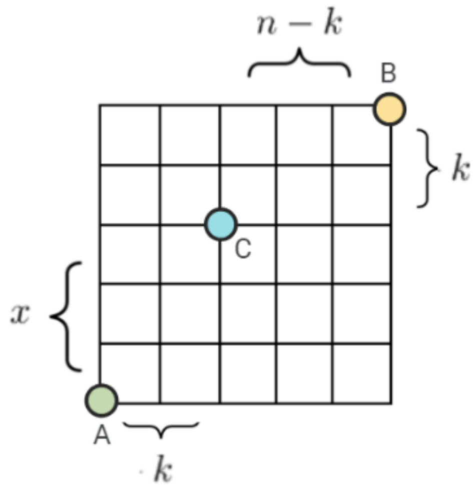


Figure 2.3.5: Grid Walk with Substituted Coordinates

Great! Now, we can compute the total number of paths it would take from C to B without going to the left or downwards. That would be, according to our discovery of Definition 2.2.4,

$$\binom{n-k+k}{k} = \binom{n}{k}.$$

Therefore, by the multiplication principle, the total number of paths to take from point A to point C and then from C to B is

$$\binom{n}{k} \times \binom{n}{k} = \binom{n}{k}^2.$$

Lastly, we just need to account for the total number of values k can take on in order to express all possible paths from A to B. The lowest it can go is 0, making point C have a x point of 0 walks. The highest it can go is n, which would make point C have a x point of n walks, being at the same horizontal point as point B.

Therefore, the summation of $\binom{n}{k}^2$ from $k = 0$ to n gives the total possible points that C can be to bridge the path from A to B into two paths from A to C and then from C to B. $\square$

This alternate proof is an amazing alternative you may use for solving these types of questions and adds to a variety of combinatorial thinking approaches and interpretations of binomial coefficients.

Let's now turn our attention to a problem where we show that double counting can also be used for proofs outside of binomial coefficient territory.

Example 2.3.4

Consider all base a positive integers ($\mathbb{Z}^+$ or $\mathbb{N} \cup \{0\}$) with at most n digits and a leading digit of 1. Use this to prove the **geometric sum** formula, where r is the common ratio and n is the number of terms in the sequence, which is non-negative. The geometric sum formula is provided below.

$$\frac{r^n - 1}{r - 1} = \sum_{k=1}^{n} r^{k-1} = 1 + r + r^2 + ... + r^{n-1}$$

Solution. For simplicity, for us to mention RHS and LHS more easily, let's write one main equation down below by removing the middle expression, as we already know that is equal to the sum on the RHS:

$$1 + r + r^2 + ... + r^{n-1} = \frac{r^n - 1}{r - 1}$$

To start, let's examine the LHS. Let's think of this as using slots once again. So our objective is to count how many possible cases there are. Well, first off, there are n possible lengths of sequences that we can create. If we try to examine the choices for each possible length, then we can use the multiplication principle and multiply them. Think of this as using slots like we did using permutations and also when we had ordered selection with repetition. When th sequence length is 1, there would be only one possibility in which the digit starts with 1:

$$\underline{\ 1\ } \text{ choice}$$

But what if we had a sequence length of 2? Then we again need digits that start with 1, so that is 1 choice for the first slot and then the second slot will allow for a choices, which are the numbers 0, 1, . . . , $a-1$. This is because all a based numbers have digits that range from 0 to $a-1$, just like we are using a base 10 representation and our numbers range from 0 to 9 for every digit "slot".

$$\underline{\ 1\ } \times \underline{\ \mathrm{a}\ } \text{ choices}$$

Then for a sequence of 3, we see something similar happening. We have again 1 choice for the first slot, which needs to be 1, but then the next two slots both have a many choices or numbers that we can put in that slot:

$$\underline{\ 1\ } \times \underline{\ \mathrm{a}\ } \times \underline{\ \mathrm{a}\ } \text{ choices}$$

Yes, you guessed it! This keeps going until we reach the maximum number slot, n. So this obtains the LHS expression when we add all the choices for each possible n, which ranges from 1 to n.

$$\begin{aligned} &1 + a + a \times a + a \times a \times a + \cdots + \underbrace{a \times a \times \ldots}_{n \text{ times}} \\ &= 1 + a + a^2 + a^3 + \cdots + a^n \end{aligned}$$

Next, we can look at the RHS. The RHS has two parts, the numerator $(a^n - 1)$ and the denominator $(a - 1)$. What do each of these represent? Well, let's see. When I see a division of two numbers in combinatorics, it reminds me of dividing some number out of the total to find the number that satisfies the question. This might just be it.

a^n is a sequence of n numbers, in which each number can basically be thought of as slots and we have a choices:

$$\underbrace{\underline{\text{ a }} \times \underline{\text{ a }} \times \underline{\text{ a }} \times \ldots \times \underline{\text{ a }}}_{n \text{ times}}$$

Realize that this does indeed account for all the possible ordered sequences, since if we started with zeros then that would represent some sequence with less than n as its length, so it accounts for all lengths. Next, the subtraction of 1 makes sure that all the numbers are positive, as the question stated that we are looking for all positive numbers, which also means non-zero. Therefore, the 1 case that we are subtracting is when all slots are equal to 0, making 0. This numerator represents all the positive cases.

The denominator is $a - 1$, which seems to be dividing something out of the total positive cases. Additionally, this must apply to any length case. Take a sequence of length x, then the starting number has $a - 1$ choices from 1 to $a - 1$ since if we had 0, that would be representing a different sequence length of $< x$.

Aha! So the bottom essentially deals with the starting digit of some sequence with length x, dividing the cases where the initial digit is any number other than 1. Since there are $a-1$ digits possible for the starting digit, then dividing all cases by $a-1$ yields the total number of cases that only have 1, as all of those $a-1$ numbers as starting digits should have the same number of cases due to no other restrictions or biases of numbers.

Therefore, both the RHS and the LHS of the geometric sum formula counts the same number of cases, but in different ways: The RHS sums all the possible cases in individual cases split by sequence length ranging from 1 to n where the starting digit is 1, while the LHS divides from the total possible positive cases to ensure the starting digit is 1. Since we achieve the same result by double counting, we have thus proven

$$\boxed{1 + r + r^2 + ... + r^{n-1} = \frac{r^n - 1}{r - 1}} \text{(Formula 1).}$$

†

I want to now introduce a very famous identity in combinatorics and double counting known as the Hockey-Stick Identity.

Theorem 2.3.5 (The Hockey-Stick Identity)

In combinatorics, the Hockey-Stick Identity (also known as the Christmas Stocking Theorem) depicts a unique relationship in Pascal's Triangle. Instead of counting the sum of a row, this theorem describes the sum of a diagonal of length n as a "hockey stick" that is equivalent to the last part of the stick. The identity can be expressed as:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

This theorem is quite interesting, and you may have already noticed this relationship, but have a look at the normal Pascal's Triangle, with 8 rows:

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

Figure 2.3.6: Normal 8-Row Pascal's Triangle

Now, take any diagonal strip that you would like. Some patterns that may jump out at you are first that there are two main diagonals of 1's, then two diagonals of an arithmetic sequence. But then, notice that by taking the difference of two numbers on a diagonal, you will find a pattern in which it yields another number in a different, adjacent diagonal. Take the sequence (1, 4, 10, 20) for example:

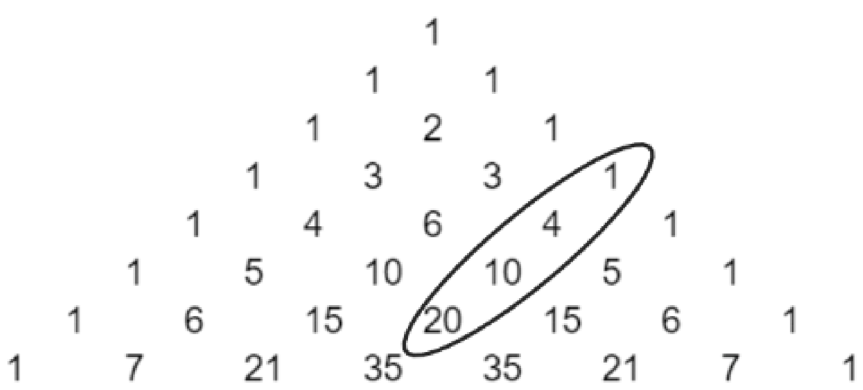


Figure 2.3.7: One Diagonal of Pascal's Triangle

What is the sequence of differences? Well $4 - 1$ is 3. $10 - 4$ is 6. $20 - 10$ is 10. So the sequence of differences that this diagonal forms is $(3, 6, 10)$. This is due to the creation of Pascal's Triangle using Pascal's Identity. However, let's now take the sum of our sequence. $1 + 4 + 10 + 20 = 35$. This is equal to both numbers below 20, shown in yellow, and our sequence is also shown below, in cyan (image on next page):

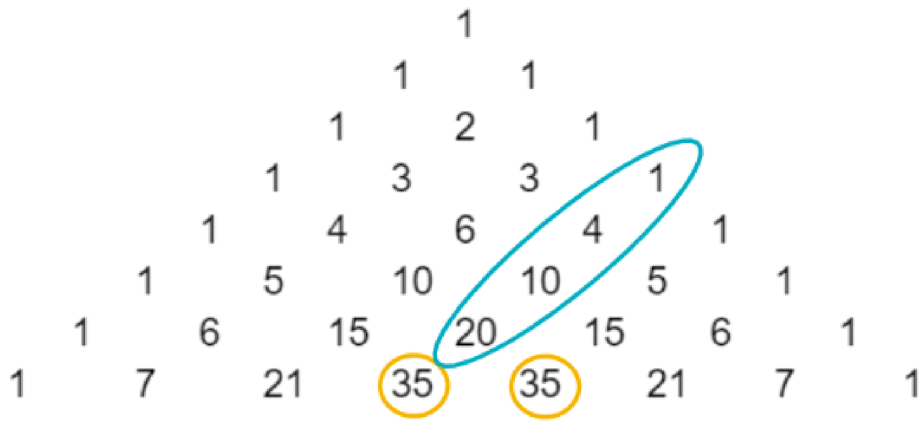


Figure 2.3.8: Sum of Circled Numbers of Pascal's Triangle

Pretty cool right? Now what about $1 + 4 + 10$? That is 15, the number right below 10, but to the right, as shown below:

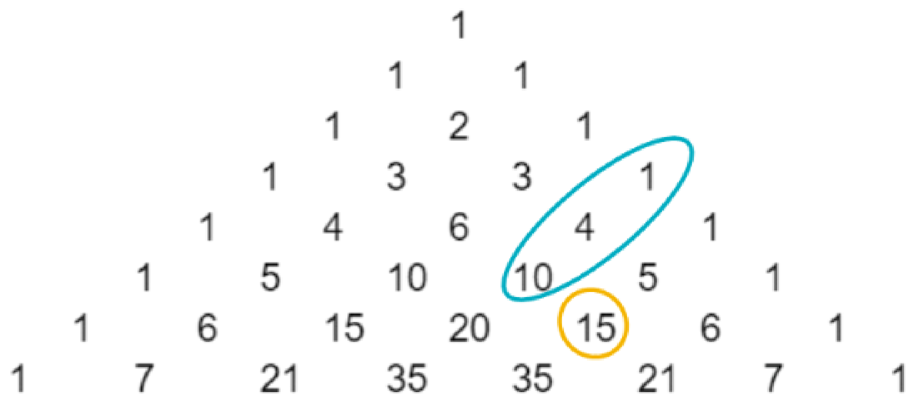


Figure 2.3.9: Another Sum of Pascal's Triangle

From this image, you can see that the $1 + 4$ also makes the 5 below and to the right of the last number in the sequence, 4. Look at our difference sequence too, we had evaluated that to be the diagonal (1, 3, 6, 10). The sum of these numbers is equal to $1 + 3 + 6 + 10 = 20$. Aha! The sum of any diagonal lies on the bottom right of the last number in the diagonal, as we see the number 20 on the bottom right of the last number in the sequence, 10.

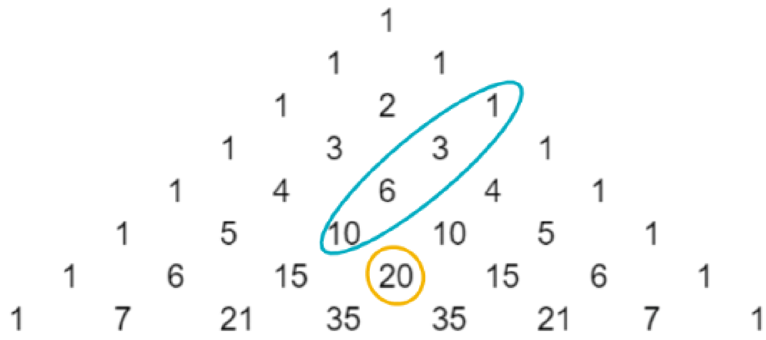


Figure 2.3.10: Different Diagonal Sum of Pascal's Triangle

But of course, this is for diagonals that are cannot start further back. For example, if you take the 1 in the cyan circled region and take the sum of that with the other 1's going down, you will not obtain the correct answer because you can go back further to start at the 1 on the top, which then would make our conjecture correct.

Now, how do we express or build this conjecture in terms of binomial coefficients? Consider the binomial coefficient version of Pascal's Triangle, similar to Figure 2.1.2:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & \cdots & & & & &
\end{array}
$$

Figure 2.3.11: Pascal's Triangle with Binomial Coefficients

So we know that $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} = \binom{5}{3}$. This represents the $1 + 3 + 6 = 10$ in the original Pascal's Triangle. See the bolded binomial coefficients below that represent this sequence and its sum's equivalent binomial coefficient:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{\mathbf{2}}{\mathbf{2}} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{\mathbf{3}}{\mathbf{2}} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{\mathbf{4}}{\mathbf{2}} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{\mathbf{5}}{\mathbf{3}} \quad \binom{5}{4} \quad \binom{5}{5} \\
\cdots
\end{array}$$

Figure 2.3.12: Hockey Stick

Our conjecture, then, can be formulated by replacing our starting spot of $\binom{2}{2}$ as $\binom{r}{r}$. We are then summing

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} = \binom{r+3}{r+1}.$$

Replacing $r+2$ with n, and adding dots to indicate flexibility of any length diagonal starting furthest back, we obtain the Hockey-Stick Identity:

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

In other words, we get this as our conjecture:

$$\boxed{\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}} \text{(Theorem 2)}.$$

An equivalent expression of this identity can also be constructed through Pascal's Triangle's symmetry to form a reverse hockey stick:

$$\begin{array}{c}
\binom{0}{0} \\
\binom{\mathbf{1}}{\mathbf{0}} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{\mathbf{2}}{\mathbf{1}} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{\mathbf{3}}{\mathbf{2}} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{\mathbf{4}}{\mathbf{2}} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\cdots
\end{array}$$

Figure 2.3.13: Inverted Hockey Stick

where the $\binom{4}{2}$ is the last part of the stick. This can be represented like so:

$$\binom{1}{0} + \binom{2}{1} + \binom{3}{2} = \binom{4}{2}.$$

In number form, this is $1 + 2 + 3 = 6$, which is true. Now we can replace 0 with r to obtain the following:

$$\binom{r+1}{0} + \binom{r+2}{1} + \binom{r+3}{2} = \binom{r+n+2}{2}.$$

or more generally,

$$\binom{r+1}{0} + \binom{r+2}{1} + \cdots + \binom{r+n+1}{n} = \binom{r+n+2}{2},$$

$$\boxed{\sum_{i=0}^{n} \binom{r+k+1}{k} = \binom{r+n+2}{n}} \text{(Formula 3).}$$

A cool picture showing two Hockey-Sticks can be drawn, with each bubble representing a sequence and a last-part-of-the-stick sum:

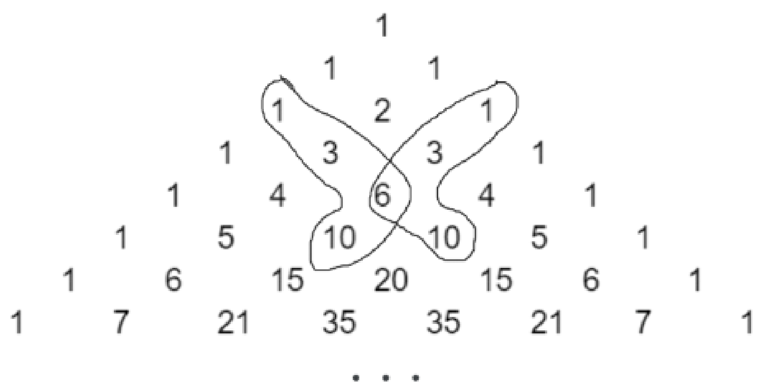


Figure 2.3.14: Double Hockey Sticks

Now begs the question, how do we prove it? Let's prove the original Hockey-Stick Identity, as it is easier to prove than Formula 3 while being equivalent (basically, a substitution can be made to Formula 2 to achieve Formula 3).

Example 2.3.6

Prove the Hockey-Stick Identity through double counting.

Solution. Recall Formula 2:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

On the RHS, we are choosing subsets of size $r+1$ out of a set of $n+1$ elements, $S = \{1, 2, \ldots, n+1\}$.

On the LHS, it is essentially the same thing, but we are splitting it into cases. The hard thing to realize is that the k can actually represent the maximum number in the subset. If we treat it that way, then we will see that everything works out just right.

Remember, we are taking $r+1$ items from the set $S = \{1, 2, \ldots, n+1\}$. In this subset that we pick, there will always be a maximum number due to the distinctness of items in a set. Let's call this max item $max(S)$. This maximum item will always be fixed within a single selection. We also know that this maximum item will have the range of $r+1 \leq max(s) \leq n+1$.

This is because we have $r+1$ distinct items that will be picked in total, so naturally—and we will introduce a principle in a future chapter that can be referred to when reasoning this way—the minimum it can take is $r+1$ if we had the chosen subset as $\{1, 2, \ldots, r+1\}$, and the maximum will be greater for any other subset, up to a maximum when the value $n+1$ is in the chosen subset.

Therefore, we can treat this maximum item as already picked since it is still arbitrary. If this item has already been picked or is "fixed," then we must still make r other selections. Additionally, this would mean that there are $max(S) - 1$ items $(1, 2, \ldots, max(S) - 1)$ we can choose r items from. This is because $max(S)$ is defined as the maximum number, so all other items in the set must be numbers lower than it, and that is the only requirement implaced on the other items, meaning we are choosing from $\{1, 2, \ldots, max(S) - 1\}$. This is equal to $\binom{max(S)-1}{r}$ many possible choices.

Let's take a look at our LHS again. This kind of matches the summand expression, but not really. What if we let $max(S) - 1 = k$? Not a bad idea! Let's see why.

Our last step is to account for the variations that the maximum value may still take. In order to account for this arbitrariness of the maximum value, we must sum over all possible values of this item, where each possible value of $max(S)$ is equal to $\binom{k}{r}$. $max(S)$ can only take values from $r+1$ to $n+1$ as we have previously defined it. Let us rewrite the inequality and see what happens when we substitute $max(S) = k+1$, which was previously defined from $max(S) - 1 = k$:

$$r + 1 \leq max(S) \leq n + 1$$

Substitute $max(S) = k + 1$,

$$r + 1 \leq k + 1 \leq n + 1$$

Subtract 1 on all three sides,

$$r \leq k \leq n.$$

Aha! Now we know all the possible values for k ranges from k to n, we can sum that up, and that is precisely equal to the summation on the LHS. So the LHS becomes:

$$\sum_{k=r}^{n} \binom{k}{r}$$

Therefore, we have proved that

$$\boxed{\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}}.$$

†

Amazing. There are also other theorems that are constructed based off of this Hockey-Stick Identity. Some of these theorems will be left for you to explore and prove using double counting in the exercise problem set.

Finally, I want to end by showing you a problem that can be very simply solved by double counting. Feel free to give this a try before I go on.

Example 2.3.7

Named after the French mathematician Alexandre-Théophile Vandermonde, Vandermonde's identity is a fundamental result in combinatorics. Prove this identity using double counting. Vandermonde's identity is expressed by the equation below, where m, n, and r are any non-negative integers:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Proof. Suppose a committee consists of m men and n women. A subcommittee of r members can be created in the following amount of ways:

$$\binom{m+n}{r}.$$

This is also represents the total number of ways to create two subcommittees of some fixed k men and fixed $r-k$ women to create the same committee of size r, given by

$$\binom{m}{k}\binom{n}{r-k}.$$

Thus, the sum of total possible values for k—between having a subcommittee of no men and a subcommittee of all r members being men—obtains the total number of ways to create two subcommittees of a total of r men and women:

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

□

Cool! As you may recall, this is quite similar to Example 2.3.3. In fact, this is the exact same if it was true that $r = m = n$.

We have finally reached the end of the chapter. Hopefully by now, you have learned three ways of proving these combinatorial identities: Proof by Induction, Algebraic Manipulation, and Double Counting. Algebraic manipulation can also be used in a different way rather than directly substituting. For Vandermonde's Identity, for example, you cannot just substitute the formula because the summation is arbitrary. Therefore, you may want to consider some clever algebraic tricks in order to derive this formula using some other formula. In conclusion, double counting not only offers a fundamental method of proof, but it is also the heart of combinatorial thinking. A problem set is present in the exercise below.

These questions will challenge you to think more combinatorially and are overall good practice for double counting in the future. While most problems will ask you to prove using double counting, feel free to try using prove algebraic manipulation or proof by induction for fun. Again, for double counting, you may choose to use either the subset, committee methods, or any other equivalent methods, it it all up to preference. Good luck!

Youtube Lectures 2.3

1. Proving the Binomial Theorem: A Combinatorial Proof

2. Proving Pascal's Symmetry using Double Counting

3. Proving Vandermonde's Identity: Double Counting Technique

Exercises 2.3

(next page)

Exercise 2.3.1

The Hockey-Stick Identity has an equivalent form by starting at 0 instead of r, since when $k > r$, the binomial coefficient is equal to 0. Use this insight to prove the Hockey-Stick Identity by induction.

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{k+1}{r+1} \text{ (Formula 2)}$$

Exercise 2.3.2

Is the following equation true? Use combinatorial argument. Then, confirm through algebra or induction.

$$\binom{n+1}{2} + \binom{n}{2} = n^2$$

Exercise 2.3.3

Use exercise 1 and 2 to find a closed form expression for

$$\sum_{k=1}^{n} k^2.$$

Exercise 2.3.4

Use double counting to prove that

$$\binom{k}{2} + \binom{n-k}{2} + k(n-k) = \binom{n}{2}.$$

Exercise 2.3.5

Use a combinatorial argument to prove that the following expression is even. You may want to consider Example 2.3.7 and Definition 2.2.1.

$$\binom{2n}{n} = 0 \pmod 2.$$

Exercise 2.3.6

Prove that

$$\sum_{k=0}^{n} \binom{k}{r} k = \binom{n+1}{r+1} n - \binom{n+1}{r+2} \text{ for } n \geq 0, k \geq 0.$$

Consider the following for Exercise 2.3.7, 2.3.8, 2.3.9, 2.3.10, and 2.3.11.

The Binomial Theorem (Theorem 2.2.1) gives the following when $y = 1$:

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Use this to find a simple expression for each of the following expressions:

Exercise 2.3.7

$$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

Exercise 2.3.8

$$\sum_{k=1}^{n} k\binom{n}{k}$$

Exercise 2.3.9

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}x^{k+1}.$$

Exercise 2.3.10

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$$

Exercise 2.3.11

$$\sum_{k=0}^{n} (-1)^k\frac{1}{k+1}\binom{n}{k}$$

Exercise 2.3.12

The product of two polynomials of degrees m and n is generally given by

$$\left(\sum_{i=0}^{m} a_i x^i\right)\left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{r=0}^{m+n}\left(\sum_{k=0}^{r} a_k b_{r-k}\right)x^r,$$

where, by convention, $a_i = 0$ for all integers that $i > m$ and $b_j = 0$ for all integers $j > n$.

Prove Vandermonde's Identity (Theorem 2.3.4) using algebraic manipulation and substitution.

Hint: Using the Binomial Theorem equation given in the previous set of exercises, formulate an equation for $(1+x)^{m+n}$.

Exercise 2.3.13

Using double counting, prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Exercise 2.3.14

n teams participate in a tournament where there are no draws—each match produces one loser and one winner. The total number of wins and losses of team i are denoted by W_i and L_i, respectively. Use combinatorial thinking to prove that

$$W_1 + W_2 + \cdots + W_n = L_1 + L_2 + \cdots + L_n.$$

Exercise 2.3.15

A committee consists of a certain number of members. Each member belongs to exactly three subcommittees and each subcommittee has exactly three members. Prove that the number of members is equal to the number of subcommittees.

Exercise 2.3.16

Use a combinatorial approach to prove that the following equation gives the largest number in any row of Pascal's Triangle.

$$\binom{2n}{n}.$$

Exercise 2.3.17

Prove that

$$\sum_{k=1}^{n-1} k(n-k) = \binom{n+1}{3}.$$

Exercise 2.3.18

Let $x_1 \leq x_2 \leq \ldots \leq a_n = m$ be positive integers. If b_k is the count of a_i in which $a_i \geq k$, then prove that the following equation holds.

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_m.$$

Exercise 2.3.19

Use double counting to prove the following formula:

$$\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2}.$$

Hint: Consider a row of $n+1$ dots

Exercise 2.3.20

Let $p_n(k)$ denote the number of permutations of some set $[n] = \{1, 2, 3, \ldots, n\}$. If this set has exactly k fixed points, show that the following equation holds:

$$\sum_{k=0}^{n} kp_n(k) = n!\,.$$

Exercise 2.3.21

In Exercise 2.2.6, you proved the following equation through induction. Now, use a combinatorial argument to prove that

$$\sum_{k=1}^{n} k \cdot k! = (n+1)!\, -1.$$

Exercise 2.3.22

In Exercise 2.2.5, you proved the equation below through proof by induction. Now, use double counting to prove that

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Exercise 2.3.23

15 students join a summer course. Every day, three students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Exercise 2.3.24

16 students took part in a mathematical competition where every problem was a multiple choice question with four choices. After the contest, it is found that any two students had at most one answer in common. Prove that there are at most 5 problems in the contest.

Exercise 2.3.25

During an assessment event at a school, there are i students and an odd number of teachers, denoted by j, where $j \geq 3$. Each teacher determines if each student passes or fails. Let n be the number such that for any pair of teachers, they both assign pass or both assign fail for at most n students. Then, prove that the following inequality is true:

$$\frac{n}{i} \geq \frac{j-1}{2j}.$$

Exercise 2.3.26

Let

$$S(n,m) = \binom{n}{0}(2^n-1)^m - \binom{n}{1}(2^{n-1}-1)^m + \binom{n}{2}(2^{n-2}-1)^m + \dots + (-1)^{n-1}\binom{n}{n-1}.$$

Prove that $S(n,m) = S(m,n)$.

Exercise 2.3.27

Apply double counting to prove that

$$\sum_{k=0}^{n} \binom{n}{k} k = n \cdot 2^{n-1}.$$

Then, use the Binomial Theorem and the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$ to prove the formula.

Hint: You may remove an index of a summation as long as that the summand evaluated at that index is equal to 0.

Exercise 2.3.28

Apply double counting to prove that

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1) \cdot 2^{n-2}.$$

Side note: This is a very cool problem. The combinatorial proof is much easier than proof by induction, but feel free to try that as well. You may also consider an algebraic approach using the Binomial Theorem and one step of calculus. The last method I just described is very common for solving these problems, including proving the Binomial Theorem.

I encourage you to try and develop a simple proof for this problem and/or the Binomial Theorem using the algebraic and calculus approach, where you will need to differentiate both sides to obtain your desired results. This comes from the linearity of differentiation. Formally, if $f(x) = g(x)$, then $f'(x) = g'(x)$.

Anyways, I hope that this chapter was an eye-opening experience as an introduction to combinatorics.

CHAPTER 3

Strategic Counting and Probability Theory

3.1 Complementary Counting

Let's start with a very simple problem.

Example 3.1.1

In a class of 40 students, 28 students are taking Math. How many students are not taking Math?

Solution. The number of students $|S|$ is equal to 40 and the number of students taking Math is $|M|= 28$. Therefore, the number of students that are not taking Math is equal to $|S|-|M|= 40 - 28 = 12$. †

Easy, right? The number of students who are not taking Math can be denoted by the complement of it, often like $|M^c|$, where M^c is the set of students who do not take Math.

This way of counting, to achieve what we desire by subtracting the whole by the part that complements what we want to achieve is called complementary counting.

Theorem 3.1.2 (Complementary Counting)
If S is the universal set containing all possible outcomes and A is the set of favorable outcomes, then the number of elements in A, $|A|$, is given by:

$$|A| = |S| - |A^c|,$$

where A^c represents the complement of set A. That is, all the elements in set S that are not apart of set A.

Although this theorem is quite simple and intuitive, it can be quite useful throughout tons of math problems. Here's a visual representation of this theorem:

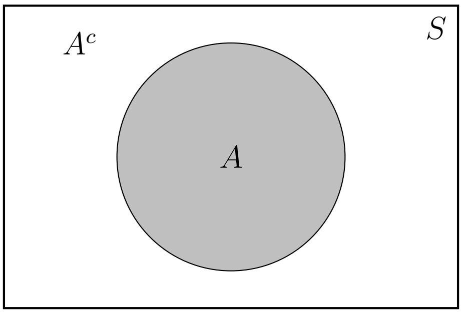


Figure 3.1.1: Complementary Sets A, A^c in Universal Set S

Example 3.1.3
Find the number of integers x where $1 \leq x \leq 1000$ and x contains at least one digit of 8.

Solution. We could break this into cases, i.e. use casework. We break them into having one, two, and three 8's. We know any number from 1 to 1000 can be expressed within 4 digits. For example, the number 684 can be expressed as 0684.

$$\underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}$$

For the case in which we have one 8, we have the following amount of choices for each digit, or "slot" (for the specific case in which the second slot is 8):

$$\underline{\ 1\ }\ \ \underline{\ 1\ }\ \ \underline{\quad}\ \ \underline{\quad}$$

Notice that the first slot was filled by 1, because the only option is going to be 0 for the first digit, making it a 3 digit number since 1000 does not contain the number 8. Additionally, the second slot is fixed, since we set it to be 8 specific to this case. The third slot has 9 options (numbers from 0 to 9 minus the one number 8), same with the fourth. So we have:

$$\underline{\ 1\ }\ \ \underline{\ 1\ }\ \ \underline{\ 9\ }\ \ \underline{\ 9\ }$$

Since 8 can either be in slot 2, slot 3, and slot 4 and it has the same choices that would be filled for the other slots (such as 8 being fixed at slot 3, there would be 9 choices in slot 2 and slot 4 similar to this case), meaning that there are 3 possible cases of one 8 where each case has 9×9 choices. Thus, we have $9 \times 9 \times 3 = \boxed{243}$ numbers that contain one 8 between 1 and 1000.

Next, we look at the cases for two 8's. Again, the first slot cannot be 1, so it must be 0, and in this instance, we want to fix two 8's. Let's try one of the instances:

$$\underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\quad}$$

We say that slot 2 and slot 3 are fixed by 8. Then, there are again 9 options for the other slots being numbers from 0 to 9, not including 8. So the last slot has 9 choices:

$$\underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\ 9\ }$$

So there are

$$1 \times 1 \times 1 \times 9 = 9$$

choices per two 8's being fixed. Now let's see how many possible rearrangements we can make of these two 8's. Remember the first slot is always 1 since 1000 doesn't work, so the 8's can only move around the second to fourth slots. Therefore, there are

$$\binom{3}{2} = \frac{3!}{(3-2)!\,2!} = \frac{6}{2} = 3$$

possible combinations to fix the two 8's across two of the three available slots. Since there are 3 possible combinations and each combination has 9 numbers, we have $3 \times 9 = \boxed{27}$ integers that have two 8's as their digits between 1 and 1000. Keep in mind we know these numbers are unique to one another since each case has different digits and we made sure that we do not have any possibility of repetition by limiting the available slots with 9 choices, excluding the 8 that can cause repetition.

Next, having three 8's obviously has only one possibility, and that is 888:

$$\underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\ 1\ }\quad \underline{\ 1\ }$$

There are no available slots as soon as we fixed all of the slots. Therefore, there are $1 \times 1 \times 1 \times 1 = \boxed{1}$ case in which there are three 8's for integers between 1 and 1000.

Let's now add all of these to obtain our answer:

$$243 + 27 + 1 = \boxed{271}.$$

Wew! What if I told you that complementary counting can make this 100 times easier? Let's see how many numbers are in the complementary set, that is the set of those numbers from 1 to 1000 without 8's.

Again, we have 4 slots, and 1000 (when the first slot is equal to 1) is one of the cases so we already know that, and so let's check the number of these cases for where the first slot is equal to 0:

$$\underline{\ 1\ }\quad \underline{\quad}\quad \underline{\quad}\quad \underline{\quad}$$

Well, what are the rest of the slots if we cannot have the number 8? Easy! Each slot has 9 choices of numbers ranging from 0 to 9, not including 8. So we have the following count of numbers that satisfy numbers not including 8:

$$\underline{\ 1\ } \times \underline{\ 9\ } \times \underline{\ 9\ } \times \underline{\ 9\ } = 729.$$

Now we add the case of 1000 so it will become $729 + 1 = 730$. Wait a minute! One of the cases does not work when we set each choice to be 9 choices. This is because each of the 9 choices includes 0, so there is case where all 3 slots were 0's, and since we were asked for integers ranging from 1 to 1000, 0 would not work. So we would subtract one case from 729 to obtain 728. Then, we add the case for 1000, obtaining 729.

We know the maximum amount of choices is

$$\underline{\ 1\ } \times \underline{\ 10\ } \times \underline{\ 10\ } \times \underline{\ 10\ } = 1000$$

for when slot 1 is equal to 0, but again we have the case where the other 3 slots can all be 0, which we would have to subtract from. So we have $1000 - 1 = 999$. Then, we add the case of 1000 to make it 1000.

Therefore, by complementary counting, we again find that there are $1000 - 729 = \boxed{271}$ integers from 1 to 1000 that have at least one 8 in their digits. †

Exercises 3.1

Exercise 3.1.1

How many integers from 1 to 100 are not prime?

Exercise 3.1.2

License plates have 5 letters on it and on every one of them, there will always be at least 2 letters in a row that are the same. For example, PRESS or MEETS work as license plates, while BUSES cannot be a license plate. Jake wants to find how many license plates he can create satisfying this criteria.

Exercise 3.1.3

How many positive integers less than 100 are not a multiple of five?

Exercise 3.1.4

Compute the number of four-digit positive integers such that at least one of its digits that is either a 2 or a 3.

Exercise 3.1.5

Sally is drawing seven houses. She has four crayons, but she can only color any house a single color. In how many ways can she color the seven houses if at least one pair of consecutive houses must have the same color?

Exercise 3.1.6

A company uses a unique identification system for its employees consisting of a sequence of three uppercase letters followed by a sequence of three digits. Given that each possible combination of three letters and three digits is equally likely, determine the probability that such an ID contains at least one palindrome (either a three-letter sequence or a three-digit sequence that reads the same backward as forward). The probability can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Exercise 3.1.7

Hector bought a new monster truck that takes up two spots on a parking lot. He wants to go shopping at a mall with a parking lot of 16 adjacent spaces. If all spaces are vacant, and twelve cars that each take up one spot at random parks in that parking lot, what is the probability that Hector can park in that parking lot?

3.2 The Principle of Inclusion-Exclusion

Recall that the **intersection** of sets A and B is represented by $A \cap B$ and the **union**, or the set containing all the unique elements of both sets, is given by $A \cup B$. Let's represent each by a graph, as shown below.

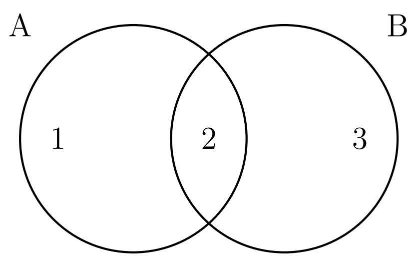


Figure 3.2.1: Intersecting Events A, B

What would the intersection be equal to?

Well, this is just given by area 2, since that is where the two events, or circles, intersect each other and share the common region 2.

Let's now recall the rule of sum, which states that for two disjoint sets A and B (meaning their intersection is equal to zero, or $A \cap B = 0$), the union is equal to 0 ($A \cup B = |A| + |B|$). This can be depicted in the figure below.

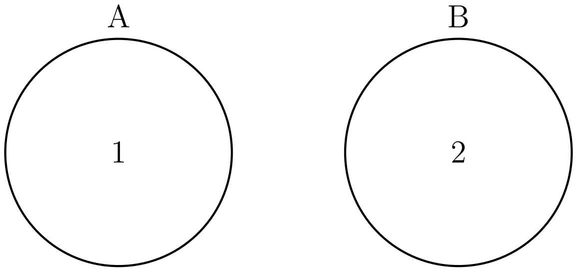
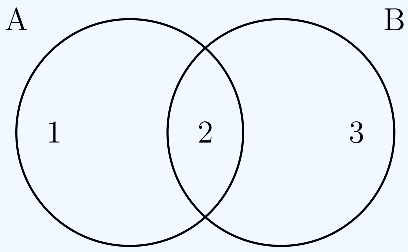


Figure 3.2.2: Disjoint Events A, B

The new area 1 would be equal to the cardinality of set A, $|A|$, while area 2 would be equal to the cardinality of set B, $|B|$. This means that the union between sets A, B is equal to the sum of areas 1 and 2, where A, B are disjoint.

Example 3.2.1

What is the union of sets $|A \cup B|$, given that $A \cap B = 9$, $|A| = 18$, and $|B| = 20$? Figure 3.2.1 is given below.



Solution. What do we know? We know that area 2, or the intersection $|A \cap B|$, summed with area 1 obtains $|A|$, while area 2 summed with area 3 gives our $|B|$.

To find the union, you may be tempted to apply the rule of sum, which would be $|A| + |B| = 18 + 20 = 38$. But notice these sets have a non-zero intersection, and therefore are not disjoint. So, we must account for the intersection.

We know the union is equal to the sum of areas 1, 2, and 3. When we add $|A| = 18$ with $|B| = 20$, we are essentially adding areas 1, 2 with areas 2, 3 that correspond to $|A|$ and $|B|$, respectively. So we would have 1 area 1, 2 area 2's, and 1 area 3. Since we want only one of each, all we have to do now is subtract by the extra area 2,

We know that area 2 is equal to $|A \cap B|$, so $|A \cup B| = |A| + |B| - |A \cap B|$. Let's plug in the numbers, given that $A \cap B = 9$, $|A| = 18$, and $|B| = 20$, we have that the union $|A \cup B| = 18 + 20 - 9 = \boxed{29}$. †

The general result that we found, $|A \cup B| = |A| + |B| - |A \cap B|$, actually applies to any sized sets A, B. Take disjoint sets, for example. By definition, these sets have intersections of 0, and so we have $|A \cup B| = |A| + |B| - 0 = |A| + |B|$. This agrees with the rule of sum.

Lemma 3.2.2

For any sets A, B, the union size $|A \cup B|$ is given by

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Great. Now take a look at the example below.

Example 3.2.3

For arbitrary sets A, B, C, find an expression for the union

$$|A \cup B \cup C|$$

when given $|A|, |B|, |C|$, the pairwise intersections $|A \cap B|, |A \cap C|, |B \cap C|$, and the intersection of all 3 sets $|A \cap B \cap C|$.

Solution. In order to formulate an expression for this, consider Figure 3.2.3 as shown below.

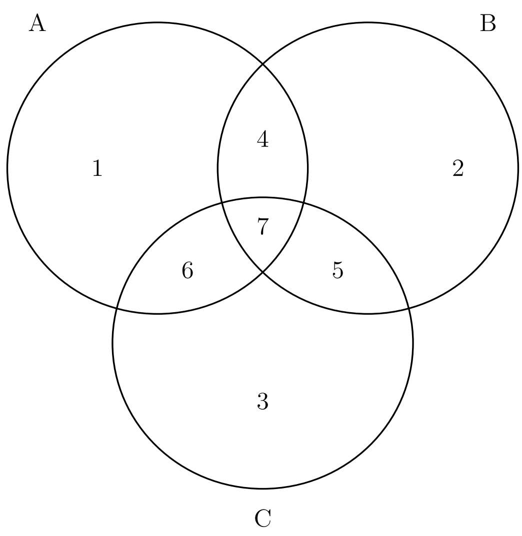


Figure 3.2.3: Events A, B, C

In this figure, area 6 corresponds to $|A \cap C|$, area 5 corresponds to $|B \cap C|$, area 4 corresponds to $|A \cap B|$, and area 7 corresponds to $|A \cap B \cap C|$.

We know that the union of A, B, C will just be all the area numbers added together. Let's try deriving the expression in two different ways: (1) starting with all 3 areas $|A|, |B|, |C|$ and subtracting out certain areas to correct our initial sum using the fact that

$$|A \cup B \cup C| = \text{area}_1 + \text{area}_2 + \cdots + \text{area}_7$$

.

Start by defining $|A|, |B|, |C|$ with the labeled areas.

$$|A| = \text{area}_1 + \text{area}_4 + \text{area}_7 + \text{area}_6,$$
$$|B| = \text{area}_2 + \text{area}_4 + \text{area}_7 + \text{area}_5,$$
$$|C| = \text{area}_3 + \text{area}_6 + \text{area}_7 + \text{area}_5.$$

So we have

$$\begin{aligned}|A|+|B|+|C| =& \text{area}_1 + \text{area}_4 + \text{area}_7 + \text{area}_6 + \text{area}_2 + \text{area}_4 + \text{area}_7 \\ &+ \text{area}_5 + \text{area}_3 + \text{area}_6 + \text{area}_7 + \text{area}_5.\end{aligned}$$

Let us simplify this down by combining like terms. We obtain:

$$|A|+|B|+|C| = \text{area}_1 + \text{area}_2 + \text{area}_3 + 2\text{area}_4 + 2\text{area}_5 + 2\text{area}_6 + 3\text{area}_7.$$

Since we know that

$$|A \cup B \cup C| = \text{area}_1 + \text{area}_2 + \cdots + \text{area}_7,$$

we can separate those areas out with the duplicate areas from the sum we obtained in order to substitute for this expression. We have that

$$\begin{aligned}|A|+|B|+|C| =& \text{area}_1 + \text{area}_2 + \text{area}_3 + \text{area}_4 + \text{area}_5 + \text{area}_6 + \text{area}_7 \\ &+ \text{area}_4 + \text{area}_5 + \text{area}_6 + 2\text{area}_7,\end{aligned}$$

or

$$|A|+|B|+|C| = |A \cap B \cap C| + \text{area}_4 + \text{area}_5 + \text{area}_6 + 2\text{area}_7.$$

Look at areas 4 (slate blue), 5 (slate blue), 6 (slate blue), and 7 (purple) highlighted in Figure 3.2.4.

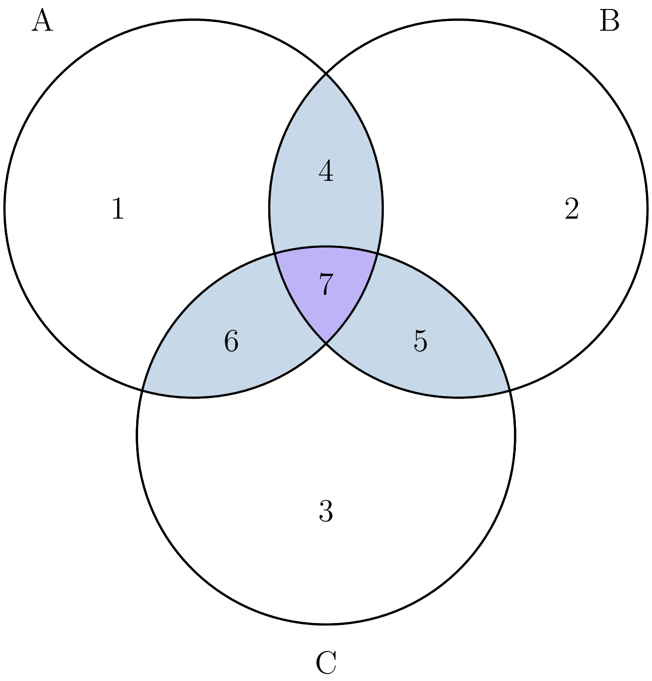


Figure 3.2.4: Regions 4, 5, 6, and 7

Recall that

$$|A \cap B| = \text{area}_4 + \text{area}_7,$$
$$|B \cap C| = \text{area}_5 + \text{area}_7,$$
$$|A \cap C| = \text{area}_6 + \text{area}_7,$$

and

$$|A \cap B \cap C| = \text{area}_7.$$

Thus, we can directly substitute into our equation after rearranging each of the equations

$$\text{area}_4 = |A \cap B| - \text{area}_7, \tag{3.1}$$
$$\text{area}_5 = |B \cap C| - \text{area}_7, \tag{3.2}$$
$$\text{area}_6 = |A \cap C| - \text{area}_7, \tag{3.3}$$
$$\text{area}_7 = |A \cap B \cap C|. \tag{3.4}$$

Let's now substitute each back in to our original equation:

$$\begin{aligned}|A|+|B|+|C| &= |A \cup B \cup C| + \text{area}_4 + \text{area}_5 + \text{area}_6 + 2\text{area}_7 \\ &= |A \cup B \cup C| + (|A \cap B| - \text{area}_7) + (|B \cap C| \\ &\quad - \text{area}_7) + (|A \cap C| - \text{area}_7) + 2|A \cap B \cap C| \\ &= |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C| \\ &\quad - 3\text{area}_7 + 2|A \cap B \cap C|.\end{aligned}$$

Substitute again for area_7:

$$\begin{aligned}|A|+|B|+|C| &= |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C| \\ &\quad - 3|A \cap B \cap C| + 2|A \cap B \cap C| \\ &= |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C| \\ &\quad - |A \cap B \cap C|.\end{aligned}$$

So we have that

$$|A|+|B|+|C| = |A \cup B \cup C| + |A \cap B| + |B \cap C| + |A \cap C| - |A \cap B \cap C|.$$

Therefore,

$$\boxed{|A \cup B \cup C| = |A|+|B|+|C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|}.$$

†

Lemma 3.2.4

For any sets A, B, C, the union size $|A \cup B \cup C|$ is given by

$$|A \cup B \cup C| = |A|+|B|+|C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

Realize that this is just manually correcting $|A|+|B|+|C|$ it by making 2 corrections to Figure 3.2.4, where the blue represents areas being covered twice and the purple represents three overlapping areas:

(1) subtracting the pairwise intersections (areas 4, 5, 6) since each of those were modeled twice initially,

(2) step 1 caused area 7 to be subtracted 3 times since each of those areas 4, 5, and 6 contained area 7, and

(3) area 7 was already added 3 times initially since each of the areas of A, B, C contained area 7, meaning we need to add area 7 back once in order to account for it.

Remember, we want to account for every of those pieces of areas once, and we are just correcting the repeated ones. This obtains the lemma we had solved.

Now, observe each lemma we solved for previously for unions 2 and 3 sets represented by $|A \cup B|$ and $|A \cup B \cup C|$, respectively, and see if you notice any patterns (lemmas are reproduced below).

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

It's an alternating sum! This may remind you of Vieta's Formulas because we are essentially adding all the individual region areas, subtracting pairwise intersection areas, adding triplewise intersections, and etc. if we wanted to conjecture this for the union between any amount of non-negative integer sets.

Theorem 3.2.5 (The Principle of Inclusion-Exclusion)

The Principle of Inclusion-Exclusion (otherwise known as PIE) gives the size of the union between any n finite sets $A_1, A_2, \ldots, A_n$:

$$\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| &= |\bigcup_{k=1}^{n} A_k| \\
&= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\
&\quad + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \\
&\quad - \cdots + (-1)^{n+1} \underbrace{|A_1 \cap A_2 \cap \ldots \cap A_n|}_{\bigcap_{k=1}^{n} A_k}
\end{aligned}$$

Now, let's prove it through proof by induction.

Proof. We will prove by induction on n sets (proving through $n+1$ sets are also a common option).

Base Case: Let $n = 1$

For $n = 1$, we have only one sets A_1 and induces only the first term of our formula. These are equal:

$$|A_1| = |A_1|$$

which is clearly true, meaning that our formula is true for the case in which $n = 1$.

Induction Hypothesis

Assume that PIE holds for any collection of $n-1$ sets. That is, for any $n-1$ sets $A_1, A_2, \ldots, A_{n-1}$, we have the following, which is essentially the PIE theorem but using $n = n-1$:

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = &\sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| \\ &+ \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k| - \ldots \\ &+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}|\end{aligned}$$

Induction Step

We need to prove that the formula also holds for n sets by utilizing our inductive hypothesis for $n-1$ sets. Consider the union of n sets $A_1 \cup A_2 \cup \ldots \cup A_n$. We can express this as the following using a distributive law of unions:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n|$$

By our inductive hypothesis for two sets that

$$|X \cup Y| = |X| + |Y| - |X \cap Y|,$$

we have:

$$\begin{aligned}|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n| = &|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| + |A_n| \\ &- |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n|\end{aligned}$$

The first term $|A_1 \cup A_2 \cup \ldots \cup A_{n-1}|$ can also be expanded using the induction hypothesis:

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| = &\left[\sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \ldots \right. \\ &\left. + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \right] + |A_n|\end{aligned}$$

$$
\begin{aligned}
&- |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= \left[\sum_{i=1}^{n-1} |A_i|\right] + |A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \\
&- |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n|
\end{aligned}
$$

Now, consider the last term $|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n|$. This can be expanded using our inductive hypothesis on $n-1$ sets on the union inside the parenthesis, then applying the distributive law of unions to add the A_n term into each term of our inductive hypothesis:

$$
\begin{aligned}
|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n| = &\sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| \\
&+ \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|
\end{aligned}
$$

Then, we substitute this into the previous equation:

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = &\left[\sum_{i=1}^{n-1} |A_i|\right] + |A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \\
&- \Bigg[\sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| + \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|\Bigg] \\
= &\left[\sum_{i=1}^{n-1} |A_i|\right] + |A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}| \\
&- \sum_{i=1}^{n-1} |A_i \cap A_n| + \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| - \ldots \\
&- (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|.
\end{aligned}
$$

Now we combine the negative:

$$
\begin{aligned}
|A_1 \cup A_2 \cup \ldots \cup A_n| = &\left[\sum_{i=1}^{n-1} |A_i|\right] + |A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \ldots \\
&+ (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_{n-1}|
\end{aligned}
$$

$$-\sum_{i=1}^{n-1}|A_i \cap A_n|+\sum_{1\le i<j\le n-1}|A_i \cap A_j \cap A_n|-\dots$$
$$+(-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|.$$

Now, we can simplify these. Let's look at each pair of simplifications.

$$\left[\sum_{i=1}^{n-1}|A_i|\right]+|A_n|=\sum_{i=1}^{n}|A_i|$$

$$-\sum_{1\le i<j\le n-1}|A_i \cap A_j|-\sum_{i=1}^{n-1}|A_i \cap A_n|=-\left[\sum_{1\le i<j\le n-1}|A_i \cap A_j|+\sum_{i=1}^{n-1}|A_i \cap A_n|\right]$$

For this expression, we can easily combine these two terms because the second term in the brackets accounts for all possible combinations of intersections with A_n, which resembles every possible intersection of A_i where i is between 1 and $n-1$ with A_j where j is equal to n. Therefore, this can be simplified by adjusting the index from $n-1$ to n:

$$-\left[\sum_{1\le i<j\le n-1}|A_i \cap A_j|+\sum_{i=1}^{n-1}|A_i \cap A_n|\right]=-\left[\sum_{1\le i<j\le n}|A_i \cap A_j|\right].$$

This also happens with

$$\sum_{1\le i<j\le n-1}|A_i \cap A_j \cap A_n|$$

where this accounts for the $k=n$ intersection of the next term where we have an intersection between A_i, A_j, and A_k where $1\le i<j<k\le n-1$ that could be brought out from the ellipsis:

$$\sum_{1\le i<j<k\le n-1}|A_i \cap A_j \cap A_k|+\sum_{1\le i<j\le n-1}|A_i \cap A_j \cap A_n|=\sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k|.$$

Every other terms in both ellipsis can be combined like this, and you may think that the last two terms added together will obtain the last term in PIE:

$$(-1)^n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}|+(-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|$$

$$? = (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|$$

But this is not true, certainly because then if we subtract

$$(-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n|$$

on both sides, then we get that the first term

$$(-1)^n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}|$$

is equal to 0. So why? Notice that they have different signs, and the previous pairs all had the same signs. Yes, the first sequence,

$$\left[\sum_{i=1}^{n-1}|A_i|\right] - \sum_{1\leq i<j\leq n-1}|A_i \cap A_j| + \dots$$
$$+ (-1)^n|A_1 \cap A_2 \cap \dots \cap A_{n-1}|$$

actually has one less index than the second sequence

$$|A_n| - \sum_{i=1}^{n-1}|A_i \cap A_n| + \sum_{1\leq i<j\leq n-1}|A_i \cap A_j \cap A_n| - \dots$$
$$+ (-1)^{n+1}|A_1 \cap A_2 \cap \dots \cap A_{n-1} \cap A_n|.$$

which means that the last matching sequence that can be combined would be the last term from the first sequence

$$(-1)^n|A_1 \cap A_2 \cap \dots \cap A_{n-1}|$$

and the second last term from the second sequence,

$$(-1)^n \sum_{1\leq i_1<i_2<\dots<i_{n-2}\leq n-1}|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_{n-2}} \cap A_n|.$$

Therefore, the left-over term would be the last term from the second sequence,

$$(-1)^{n+1}|A_1 \cap A_2 \cap \dots \cap A_n|.$$

Putting all the expressions we obtained through combining terms from each sequence, we get the inclusion-exclusion formula for n sets:

$$\begin{aligned}|A_1 \cup A_2 \cup \dots \cup A_n| = &\sum_{i=1}^{n}|A_i| - \sum_{1\leq i<j\leq n}|A_i \cap A_j| \\ &+ \sum_{1\leq i<j<k\leq n}|A_i \cap A_j \cap A_k| - \dots \\ &+ (-1)^{n+1}|A_1 \cap A_2 \cap \dots \cap A_n|\end{aligned}$$

Thus, by induction, the principle of inclusion-exclusion holds for any finite number of sets n. We have proved that the base case where $n = 1$ is true, and using an inductive hypothesis of the $n = n - 1$ version of PIE, we proved that it also satisfies the normal $n = n$ PIE, meaning that PIE works for $n = 2$, then 3, then 4 since we increment by 1, and then it extends all the way to infinity, proving PIE for all positive integer n. $\square$

Now let's apply this principle to an example.

Example 3.2.6

How many integers from 1 to 100 are divisible by 6 or 10?

Solution. Easily enough, we can identify this as a use of PIE. Let set $x = \{1, 2, \ldots, 100\}$ since we want any number divisible by 6 OR 10, that means we want the union of the set of numbers $y \subset x$, where $y_k \% 6 = 0$, and the set of numbers $z \subset x$, where $z_i \% 10 = 0$. We can see that there are going to be some numbers overlapping between sets y, z, so we have to account for that overlap. These two sets can be represented by Figure 3.2.5 below:

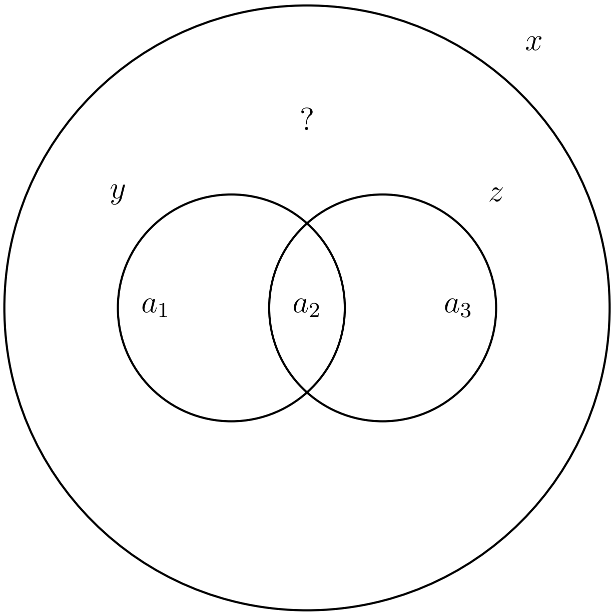


Figure 3.2.5: Sets y, z inside the set x ($|x| = 100$)

Notice that $? = 100 - (|a_1| + |a_2| + |a_3|)$. Our goal is to find $|a_1| + |a_2| + |a_3|$. We know that

$$|a_1| + |a_2| = |y|$$

and that

$$|a_2| + |a_3| = |z|.$$

Now we can just find $|y|$ and $|z|$ individually:

$$|y| = \lfloor \frac{100}{6} \rfloor = 16$$

$$|z| = \lfloor \frac{100}{10} \rfloor = 10$$

Therefore, we have the two equations:

$$|a_1| + |a_2| = 16,$$

$$|a_2|+|a_3|=10.$$

Let's add both sides of the equations:

$$|a_1|+|a_2|+|a_2|+|a_3|=26,$$

$$|a_1|+|a_2|+|a_3|=26-|a_2|.$$

Subsequent to this calculation, our job is now to find $|a_2|$ in order to find the value of $26-|a_2|$, which gives our answer.

To find $|a_2|$, we need to find intersections between y, z, which is denoted by $y \cap z$. These numbers should be divisible by both 6 and 10. To do this, we must not assume we do $\lfloor \frac{100}{60} \rfloor = 1$, since numbers like 30 also are divisible by both; we essentially need to find LCM and not assume that they are relatively prime.

Let's break down each of the factors in order to find LCM:

$$6 = 2 \cdot 3,$$

$$10 = 2 \cdot 5.$$

Thus, the LCM is

$$2 \cdot 3 \cdot 5 = 30.$$

So

$$|y \cap z|=|a_2|=\lfloor \frac{100}{30} \rfloor = 3.$$

Therefore,

$$|a_1|+|a_2|+|a_3|=26-3=\boxed{23}.$$

An alternate solution can be formed using PIE. Recall that the theorem goes with:

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n|=&|\bigcup_{k=1}^{n} A_k| \\ &= \sum_{i=1}^{n}|A_i| - \sum_{1\le i<j\le n}|A_i \cap A_j| + \sum_{1\le i<j<k\le n}|A_i \cap A_j \cap A_k| \\ &- \cdots + (-1)^{n+1}\underbrace{|A_1 \cap A_2 \cap \ldots \cap A_n|}_{\bigcap_{k=1}^{n} A_k}\end{aligned}$$

In this case, we have $n=2$, meaning that we have the following (i, j only have one possible case where $i=1, j=2$):

$$\begin{aligned}|A_1 \cup A_2| &= |\bigcup_{k=1}^{2} A_k| \\ &= \sum_{i=1}^{2}|A_i| - \sum_{1\le i<j\le 2}|A_i \cap A_j| \qquad = |A_1|+|A_2|-|A_1 \cap A_2|\end{aligned}$$

Let A_1 be the set of multiples of 6 and A_2 be the set of multiples of 10. We have previously found that $|A_1|= 16$, $|A_2|= 10$, and that $|A_1 \cap A_2|= 3$, so we again have

$$16 + 10 - 3 = \boxed{23}.$$

The figure can now be completed.

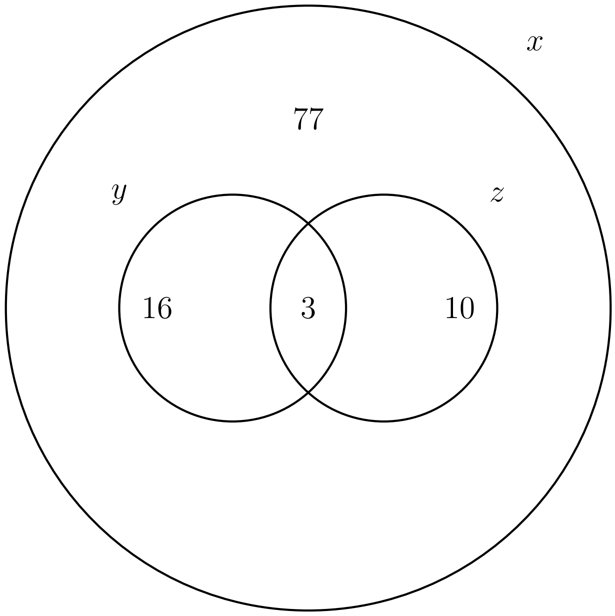


Figure 3.2.6: Sets y, z inside the set x ($|x|= 100$)

†

Let's try another example.

Example 3.2.7

In a survey of 270 college students, it is found that 64 like brussel sprouts, 94 like broccoli, 58 like cauliflower, 26 like both brussel sprouts and broccoli, 28 like both brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?

Solution. By PIE where $n = 3$ representing each vegetable (A_1, A_2, A_3 are the sets of brussel sprouts, broccoli, and cauliflower, respectively), we have the union (# of students who like any of those three vegetables)

$$\begin{aligned}
|\bigcup_{k=1}^{3} A_k| &= \sum_{i=1}^{3} |A_i| - \sum_{1 \leq i < j \leq 3} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 3} |A_i \cap A_j \cap A_k| \\
&= |A_1|+|A_2|+|A_3|- [|A_1 \cap A_2|+|A_2 \cap A_3|+|A_1 \cap A_3|] \\
&\qquad + |A_1 + A_2 + A_3|
\end{aligned}$$

We are given that

$$|A_1|= 64,$$

$$|A_2|= 94,$$

$$|A_3|= 58,$$

and

$$|A_1 + A_2|= 26,$$

$$|A_2 + A_3|= 22,$$

$$|A_1 + A_3|= 28,$$

and also that

$$|A_1 + A_2 + A_3|= 14.$$

So we have that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3|= |A_1|+|A_2|+|A_3| - [|A_1 \cap A_2|+|A_2 \cap A_3|+|A_1 \cap A_3|] \\ + |A_1 + A_2 + A_3|\end{aligned}$$

and substituting in our known values, we obtain:

$$|A_1 \cup A_2 \cup A_3|= 64 + 94 + 58 - [26 + 22 + 28] + 14 = 230 - [76] = 154$$

Therefore, there are 154 students who likes either brussels sprouts (BS), broccoli (B), or cauliflower (C). Therefore, by complementary counting, there are $270 - 154 = \boxed{116}$ students who do not like any of those aforementioned vegetables. Figure 3.2.7 shows this situation.

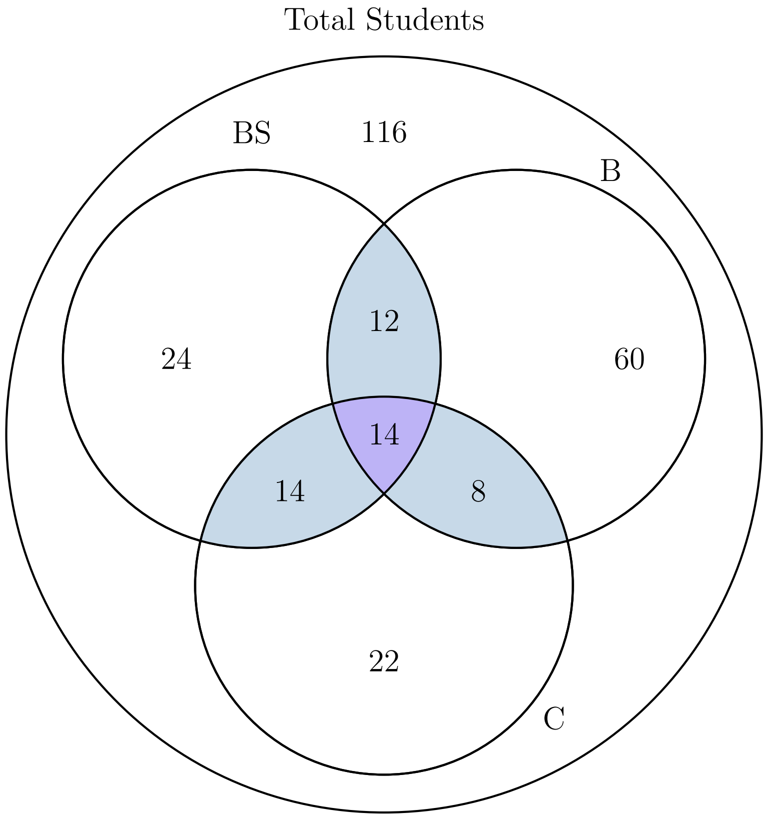


Figure 3.2.7: 270 Students' Preferences Between 3 Vegetables

†

We just used complementary counting along with PIE. PIE is often used to count situations like these where we want to find instances in which no conditions apply.

Formally, we can describe it like so below:

Theorem 3.2.8 (Complementary Form of PIE)

Let $A_1, A_2, \ldots, A_n$ be any n finite sets. The Complementary Form of the Principle of Inclusion-Exclusion (PIE) states that the cardinality of the union of these sets can be determined by subtracting the cardinality of the intersection of their complements from the total size of the universal set U. That is,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |U| - |A_1^c \cap A_2^c \cap \ldots \cap A_n^c|$$

where $A_i^c = U - A_i$ denotes the complement of A_i with respect to the universal set U.

Exercises 3.2

Exercise 3.2.1

A library has a collection of 200 books.

Of these, 90 are fiction, 80 are non-fiction, and 60 are science books. There are 40 books that are both fiction and non-fiction, 30 that are both fiction and science, and 20 that are both non-fiction and science. If 10 books belong to all three categories, how many books belong to exactly one category?

Exercise 3.2.2

A multiset is a set that allows repetition of elements. How many subsets of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c\}$$

have size 7?

Hint: This is the same as asking for the number of solutions to the equation

$$x_1 + x_2 + x_3 = 7$$

where $0 \leq x_1 \leq 2, 0 \leq x_2 \leq 3$, and $0 \leq x_3 \leq 4$.

Exercise 3.2.3

Find the number of solutions to the following equation:

$$x_1 + x_2 + x_3 + x_4 = 25,$$

where $0 \leq x_i \leq 10$.

Exercise 3.2.4

List all the 6 solutions to the restricted equation given in Exercise 3.2.2, and list all of the corresponding 6 submultisets.

Exercise 3.2.5

Find the number of integer solutions to the following equation:

$$x_1 + x_2 + x_3 + x_4 = 25,$$

where $1 \leq x_1 \leq 6$, $2 \leq x_2 \leq 8$, $0 \leq x_3 \leq 8$, and $5 \leq x_4 \leq 9$.

Exercise 3.2.6
Find the number of submultisets of the multiset

$$\{25 \cdot a, 25 \cdot b, 25 \cdot c, 25 \cdot d\}$$

of size 80, where a, b, c, d are integers.

3.3 Injections, Surjections, and Bijections

Remember that a function $f : A \to B$ is defined as a mapping the function f from one set A to another set B. The first set is called the domain and the second set is called the co-domain, where for every element in the domain, a function assigns each value in the domain to a unique value in the co-domain. In algebra class, you probably have come across what's known as a one-to-one function. A one-to-one function is a function in which the mapping from the domain to the co-domain does not map an element in the co-domain twice or more to the same element in the domain. This is otherwise known as an **injection**.

Definition 3.3.1 (Injectivity of a Function)
A function is injective (or one-to-one) if it passes the horizontal line test; that is, for every distinct pair of elements x_1 and x_2 in the domain, their corresponding images in the co-domain are also distinct. Formally, given a function $f : A \to B$, for all $x_1, x_2 \in A$, where $x_1 \mapsto f(x_1)$ and $x_2 \mapsto f(x_2)$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2,$$

or equivalently,

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

An example of a non-injective mapping is shown below in Figure 3.3.1:

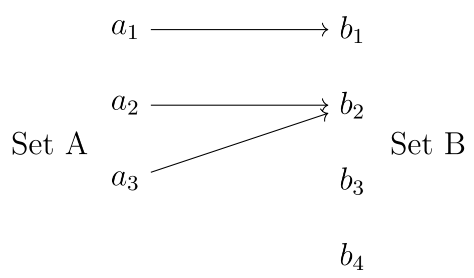


Figure 3.3.1: Non-injective Mapping

This is not an injection because two or more elements in the domain map to the same element in the co-domain: elements a_2 and a_3 map to b_2. A one-to-one function is a mapping in which there are at most one element in the co-domain mapped to by an element in the domain, such as the one below shown in Figure 3.3.2:

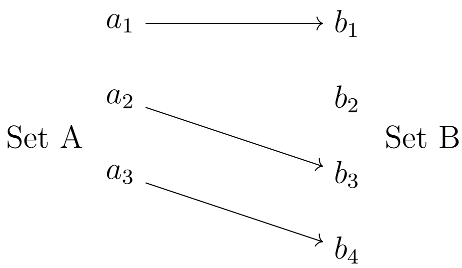


Figure 3.3.2: Injective Mapping

An onto function, on the other hand, is a mapping in which every element in the co-domain is mapped onto by an element in the domain at least once.

Definition 3.3.2 (Surjectivity of a Function)
A function $f : X \to Y$ is surjective (or onto) if for every element $y \in Y$, there exists at least one element $x \in X$ such that $f(x) = y$. In other words, the range of the function is equal to the co-domain:

$$\text{Range}(f) = Y, \quad \forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

Note that a range of a function is different than the concept of co-domain. Co-domain defines the set of all possible elements to be mapped onto, while the range describes the elements that are being mapped onto by one or more elements of the domain. An example of a function that is not surjective is shown in Figure 3.3.3 below:

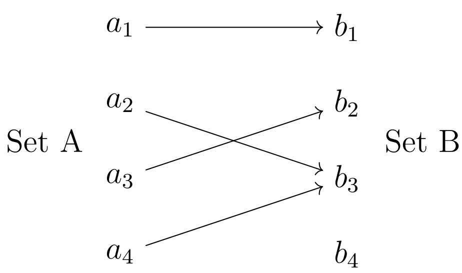


Figure 3.3.3: Non-surjective Mapping

This mapping was not surjective because it is clear that an element in the co-domain is not being mapped onto (b_4), and therefore, this is not an onto function. Instead, a surjection can be shown below in Figure 3.3.4, where we see that every single element in the domain is being mapped onto and the range is equal to the co-domain.

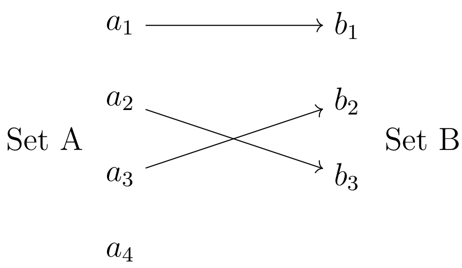


Figure 3.3.4: Surjective Mapping

Next, we define a function to be **bijective** if they are an onto (surjective) AND a one-to-one (injective) function. Bijectivity requires a function to be fully one-to-one: for a function $f : A \to B$, since injectivity limits an element in the co-domain $y \in B$ to be mapped to by at most one time and surjectivity restricts an element in the co-domain $y \in B$ to be mapped onto by at least one time, then a bijection is a function in which an element in the co-domain is mapped to by exactly one element in the domain.

Definition 3.3.3 (Bijectivity of a Function)

A function $f : A \to B$ is called a bijection if for every $y \in B$, there exists a unique $x \in A$ such that $f(x) = y$. In other words, f is bijective if the function is:

- injective: $\forall x_1, x_2 \in A,\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, and
- surjective: $\forall y \in B, \exists x \in A$ such that $f(x) = y$.

Thus, a function is bijective if an element, from any side set A (domain) or set B (co-domain), has a one-to-one correspondence where there is exactly one connection to another element in B. This invertibility must always exist in a bijection, where a bijective function $f^{-1} : B \to A$ follows this rule:

$$f^{-1}(y) := x \text{ if } f(x) = y.$$

A valid bijection is shown below in Figure 3.3.5, as every element in the domain maps to exactly one element in the co-domain:

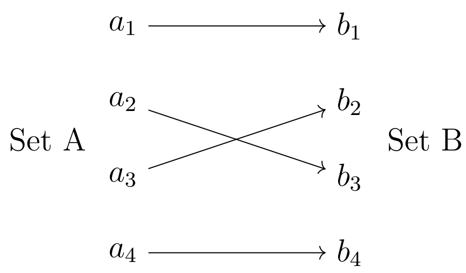


Figure 3.3.5: Bijective Mapping

A pre-image of sets is defined by

$$f^{-1}(y) = \{x \in X | f(x) = y\}.$$

Then, is it feasible to say that the pre-image rule for bijections maintains that the cardinality of the domain must be equal to the cardinality of the co-domain? (This was shown in Figure 3.3.5, where there were four elements for both sides)

Proposition 3.3.4

If X, Y are finite and $|X| = |Y|$, then a function $f : X \to Y$ is bijective, injective, and surjective.

Proof. By definition, a one-to-one function follows the rule

$$|f^{-1}(y)| \leq 1 \forall y \in Y.$$

On the other hand, an onto function is defined by

$$|f^{-1}(y)| \geq 1 \forall y \in Y.$$

Thus, since a bijection describes the intersection of the rules of a surjective and injective function, a bijective function can be defined by

$$|f^{-1}(y)| = 1 \forall y \in Y.$$

Therefore, for finite sets X, Y, there is a bijection if and only if the number of elements for each set are the same:

$$\text{Bijection } f : X \to Y \iff |X| = |Y|.$$

$\square$

What's the difference between double counting and bijective counting? Essentially, double counting counts the same set in two different, equivalent ways such as using two distinct formulas/perspectives while bijective counting establishes a function between the two sets with preserved cardinality by constructing a bijection between the two sets and ensuring there is a one-to-one correspondence between the two sets. Take a look below at diagram representations that depict the process of each (Figure 3.3.6 shows the method of double counting while Figure 3.3.7 displays the technique of bijective counting):

$$\begin{tikzcd} & S(n) \arrow[dl] \arrow[dr] & \\ g(n) \arrow[rr, equal] & & f(n) \end{tikzcd}$$

Figure 3.3.6: Double Counting

$$\begin{tikzcd} A(n) \arrow[r, shift left] \arrow[d] & B(n) \arrow[l, shift left] \arrow[d] \\ g(n) \arrow[r, equal] & f(n) \end{tikzcd}$$

Figure 3.3.7: Bijective Counting

A bijective proof is particularly useful for proving two counting methods to be equivalent by establishing the one-to-one correspondence between the sets mapped by the two methods through a bijection, showing that the two counting methods count the same number of elements and thereby are equivalent. Here is an example of a simple bijective proof of the Pascal's/binomial symmetry that we previously proved using double counting:

Example 3.3.5

Use a bijective proof to show that

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. Let $U = [n]$ (i.e. the set $[n] = \{1, 2, \ldots, n\}$) be the universal set. Next, define

$$A := \{S \subset U : |S|= k\},$$

the set containing all k-sized subsets of U. This counts

$$\binom{n}{k}$$

on the LHS. Also define

$$B := \{T \subset U : |T|= n - k\},$$

the set containing all subsets with $n - k$ elements, counting

$$\binom{n}{n-k}$$

on the RHS.

A bijection can then be constructed through complements of each of the two sets A, B. For every subset of set A, $S \subset A$, its complement is equal to

$$S^c = U \backslash S,$$

providing the set of all elements in U that are not present in subset S. Since $|U|= n$, the size of the complement of subset S (which has cardinality $|S|= k$) will be equal to

$$|S^c|= |U|-|S|= n - k.$$

Therefore, a bijection is present between the set S and its complement S^c where each subset of size k, $S \in A$, has a complement as a subset of size $n - k$, $S^c \in B$.

This also works conversely, where for every $T \in B$, the complement $T^c = U \backslash T$ will be a subset of U with size k, as T has cardinality $|T|= n - k$ and so its complement must have

$$|T^c|= |U|-|T|= n - (n - k) = k.$$

Thus, for every subset $T \in B$, $T^c \in A$. This establishes the inverse property of bijections in which the function is one-to-one for both the function and the inverse function.

This can also be shown through the involutive property of the complement operation, meaning that when you take the complement of the complement, you will obtain the original set:

$$(S^c)^c = S$$

and

$$(T^c)^c = T.$$

This property ensures a bijection where the function is both injective and surjective, thereby establishing the bijection as a whole.

Thus, a bijection shows that the subsets from set A (with size k) have a unique correspondence to the subsets of set B (with size $n - k$), and therefore, the property of bijections states that the cardinality of those two sets are equal/are counting the same number of subsets:

$$|A| = |B|,$$

and since A, B were defined to count $\binom{n}{k}$, $\binom{n}{n-k}$, respectively, then we have proved

$$\binom{n}{k} = \binom{n}{n-k}.$$

$\square$

Definition 3.3.6 (Compositions of Integers)
A composition of an integer $n \in \mathbb{Z}$ is an expression for n as an ordered sum consisting of only positive integers.

For example, all compositions of 4 are listed below:

$$1 + 1 + 1 + 1,$$

$$2 + 1 + 1,$$

$$1 + 2 + 1,$$

$$1 + 1 + 2,$$

$$2 + 2,$$

$$3 + 1,$$

$$1 + 3,$$

$$4.$$

Thus, there are 8 total compositions of 4. A question that may arise then is whether or not the number of compositions follows a trend of geometric sequence with a common ratio of 2.

Proposition 3.3.7
Let $n \in \mathbb{Z}$. Then, the number of compositions of n, $C(n)$, is given by the piecewise function below:

$$C(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ 2^{n-1} & \text{if } n > 0. \end{cases}$$

Proof. Given a number n, consider the subset $D = \{+_1, +_2, \ldots, +_n\}$, where D contains $n-1$ plus signs. There exists a bijection between the number of compositions of n and the number of subsets of D. The bijection is established as follows:

Let $S \subseteq D$ be a subset of D. The cardinality of S, $|S|$, corresponds to the number of plus signs that are not present or evaluated in the composition sum containing only 1's. The subset S indicates where summations of 1 are evaluated in the sequence of n ones. In other words, the number of plus signs not in S ($|D \backslash S|$) corresponds to the number of integers in a composition (specifically, $|D \backslash S|+1$). Essentially, for every plus sign in $S \subseteq D$, there is one less integer starting from the longest composition of all 1's—additionally, the number of ordered compositions of that length that can be rearranged would be the same as the number of ways to choose the plus signs of D into S.

$n = 4$, for example:

- If $D = \{+_1, +_2, +_3\}$ and $S = \{+_1\}$ (where $+_1$ denotes the first plus sign in D), then this subset corresponds to the composition $2+1+1$, as the first plus sign is evaluated.
- Similarly, if $S = \{+_2\}$, then this subset maps to the composition $1+2+1$, where the second plus sign is evaluated.
- The subset $S = \{+_1, +_2\}$ maps to the composition $3+1$, which represents removing or evaluating the first two plus signs.
- The empty subset $S = \emptyset$ means no plus signs are evaluated, corresponding to the composition $1+1+1+1$.
- And so on.

This bijection works because every subset S of D uniquely determines a composition of n by controlling the positions of plus signs in the sequence of 1's, meaning each composition corresponds to a unique subset of D, and vice versa.

Thus, with a bijection between the number of plus signs being chosen from a set D containing $n-1$ plus signs established, the total number of subsets possible of that set D would also count the number of compositions (Definition of Bijections shown by Proposition 3.3.4). The total number of subsets of a set of size $|D| = n-1$ is equal to 2^{n-1}. Therefore, the total number of compositions for any integer n greater than 0 is equal to

$$2^{n-1},$$

and when $n = 0$, the Principle of Nothingness states that there is only one way to create a composition of nothing, meaning $C(0) = 1$. When $n < 0$, there is no way to use positive integers to sum into a negative integer. □

Note that an alternative proof can be constructed as shown below:

Proof. A composition of integer n can be represented by the equation

$$n = n_1 + n_2 + \cdots + n_k,$$

where $n_i \geq 1$ and k is a fixed length of the composition. In chapter 1, we found a lemma to stars and bars where the sum of integers greater than 1 is equal to

$$\binom{n-1}{k-1}.$$

Therefore, the total number of compositions is given by the sum across all possible values of k, which can range from 1 (n itself) to n (sum of n 1's):

$$C(n) = \sum_{k=1}^{n} \binom{n-1}{k-1}.$$

Let $m = n - 1$ (and $n = m - 1$) and $p = k - 1$ (and $k = p + 1$), then the summation can be simplified to:

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n-1}{k-1} &= \sum_{k=1}^{m+1} \binom{m}{k-1} && \text{(Substitute } n = m-1, m = n+1\text{)} \\
&= \sum_{p+1=1}^{m+1} \binom{m}{p} && \text{(Substitute } p = k-1, k = p+1\text{)} \\
&= \sum_{p=0}^{m} \binom{m}{p}. && \text{(Simplify; adjust upper limit to match bound difference)}
\end{aligned}$$

This is the total count of subsets of any size from 0 to m of a m-set, which is just

$$2^m = \boxed{2^{n-1}}.$$

Note that the binomial theorem expansion also counts this, as $(1 + 1)^m = 2^m$. Again, this applies for only $n > 0$ and for $n \leq 0$, we may use the same logic as the previous proof. □

Just a side note, we may prove the formula for the total number of subsets also by induction (where we previously proved by double counting in the second chapter).

Proof. Given any integer $m > 0$, we want to prove that the number of subsets that can be formed of a m-set is equal to 2^m.

Base case: For $m = 1$, there are two possible cases (using [1] as the set):

$$\{1\}, \emptyset = \{\}.$$

The formula holds for this case:

$$2^1 = 2.$$

Inductive hypothesis: We assume that the number of subsets that can be formed from a set of m elements is equal to 2^m.

Inductive step: We prove that this case is also true when for $m + 1$. Counting the number of subsets of $m+1$ is also the same as counting the total number of subsets of any size from 0 to $m+1$ from a set of size $m+2$ (e.g. $[m+2] = \{1, 2, \ldots, m, m+1, m+2\}$), which is equal to the total number of subsets of the $m+2$-set subtracted by the number of subsets containing $m+2$. The total number of subsets containing $m+2$ is exactly equal to the number of subsets that can be formed from the $m+2$-set removing $m+2$, as we can make a union of $m+2$ with every of those previous subsets and so an empty subset would become a singleton set containing only $m+2$ and the set of $m+1$ elements would become a set of $m+2$ elements containing $m+2$ ($[m+2]$). Therefore, by our inductive hypothesis, the number of subsets with $m+2$ is equal to 2^{m+1}, while the total number of subsets of $[m+2]$ is equal to 2^{m+2} by the inductive hypothesis. So this is

$$2^{m+2} - 2^{m+1} = 2^{m+1}(2-1) = \boxed{2^{m+1}}.$$

□

Exercises 3.3

Exercise 3.3.1

Consider the set of integers from 1 to 1,000,000. Each number can be classified by whether the sum of its digits is odd or even. How many more numbers have an odd sum of digits compared to an even sum of digits?

Exercise 3.3.2

Let n be a fixed positive integer. How many ways are there to represent n as the sum of k positive integers, $n = a_1 + a_2 + \cdots + a_k$, where $a_1 \leq a_2 \leq \ldots \leq a_k$? For example, if $n = 4$, there are 4 ways: 4, $2+2$, $1+1+2$, and $1+1+1+1$.

Exercise 3.3.3

Tram tickets are labeled with a six-digit number between 000000 and 999999. A ticket is considered lucky if the sum of its first three digits equals the sum of its last three digits. Additionally, it is called medium if the sum of all six digits is 27. Prove that the number of lucky tickets is equal to the number of medium tickets.

Exercise 3.3.4

An insect begins at the point $(0,0)$. Each second, if the insect is at a point (x,y), it moves to one of the neighboring points $(x-1,y-1)$, $(x-1,y+1)$, $(x+1,y-1)$, or $(x+1,y+1)$. In how many ways can the insect reach the point $(2010,2010)$ after 4020 seconds?

Exercise 3.3.5

In a chess tournament, 10 players compete in a knockout format. In the first round, Player P_{10} plays Player P_9, and the loser is ranked 10^{th}, while the winner plays Player P_8 in the next round. The loser of that match is ranked 9^{th}, and so on, until the winner plays Player P_1. The loser of the final game is ranked 2^{nd}, and the winner is ranked 1^{st}. How many different rankings of the players are possible at the end of the tournament?

Exercise 3.3.6

You may have noticed that two problems from the previous exercises test similar concepts, even though their settings differ. Prove that these two problems are actually equivalent by constructing a bijection between their solutions.

Exercise 3.3.7

A circular room is divided into four equal sections. Each section is then colored with one of four distinct colors. If a coloring can be obtained by rotating another coloring, then they are considered the same. Find the number of ways that this room can be painted.

Exercise 3.3.8

Eight points are marked on the circumference of a circle, and every pair of points is connected by a line segment. No three segments intersect at a common point within the circle. Compute the number of triangles that can be formed such that a triangle's vertices are located strictly within the interior of the circle.

Exercise 3.3.9

Consider the sequence $3^0, 3^1, 3^2, \ldots$. How many of the first 1000 positive integers can be written as the sum of distinct elements from this sequence?

Exercise 3.3.10

Prove that the number of binary sequences of length n with exactly m blocks of consecutive 1's is given by the expression $\frac{n+1}{2m+1}$.

Exercise 3.3.11

Consider an equilateral triangle with side length n, divided into smaller equilateral triangles. Define $f(n)$ to be the number of distinct paths from the top vertex of the triangle to the middle vertex of the bottom row, where each path moves either horizontally or downward along an edge. Determine $f(2005)$.

Exercise 3.3.12

Consider a triangle with the following points: $A = (0, 0)$, $W = (7, 0)$, $O = (7, 1)$, and $M = (0, 1)$. In how many ways can you tile the rectangle $AWOM$ by solely using triangles with an area of $\frac{1}{2}$ that are formed on the boundary of the rectangle.

Exercise 3.3.13

Find the number of ways integers from 1 to n can be ordered such that every integer other than the first has a difference with the previous number of at most 1.

Exercise 3.3.14

Consider an equilateral triangle with side length n, divided into smaller equilateral triangles. Let $f(n)$ denote the number of distinct parallelograms formed by the smaller triangles. If $n = 3$, then $f(3) = 45$. Find a formula for $f(n)$.

Exercise 3.3.15

If $n \geq 2$ is an integer and A_n denotes the number of non-empty subsets of the set $S = [n] = \{1, 2, \ldots, n\}$ such that the mean of all elements in S is equal to an integer. Prove that $A_n - n$ is always even.

Exercise 3.3.16

Denote a permutation of the integers $[n] = \{1, 2, \ldots, n\}$ as $a_1, a_2, \ldots, a_n$, where $n \geq 2$. Find the largest possible value of the following sum:

$$S(n) = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|.$$

Exercise 3.3.17

Consider the fractional part function $\{x\}$ defined as $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Let $f(x) = \{x\}$, and find the number of solutions to the equation $f(f(f(x))) = 17$ for $0 \leq x \leq 2020$. Find the remainder when this number is divided by 1000.

Exercise 3.3.18

A binary sequence with a length of n can be defined as the sequence $(x_1, x_2, \ldots, x_n)$ where each term x_i is either 0 or 1. Let a_n denote the number of binary sequences with length n such that it does not contain any three consecutive terms of the order $0, 1, 0$. Let b_n denote the number of binary sequences at length n that does not contain four consecutive terms that are equal to either of the sequences $0, 0, 1, 1$ or $1, 1, 0, 0$. Show that the following holds for all positive integers n:

$$b_{n+1} = 2a_n.$$

3.4 Invariants and Monovariants

Do you like playing games? One of the strategies that you can use to calculate the end behavior of a game or process is by using invariants.

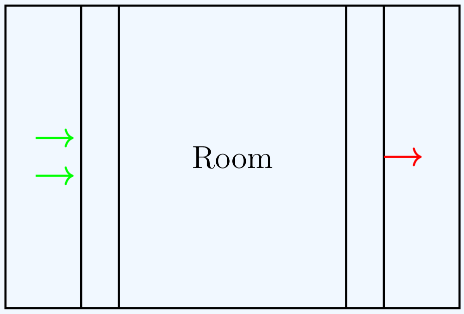
Definition 3.4.1 (Invariants)

Invariants describe a quantity or property that never changes (thus "in-variant") after a process is performed.

Using invariants can help dial down and find a common pattern between processes and help verify answers—they especially help with simplifying proofs and offer a brilliant way to understand why certain properties or outcomes occur. Let's take a look at an example.

Example 3.4.2

A room is initially empty. Every minute, either two people enter or one person leaves. Is it possible to have 298 people in the room after 3000 minutes?



Solution. To solve this problem, it is important to establish or spot an invariant in the process. The process yields either -1 people (removing one person) in the room or 2 extra people in the room. Thus, we inspect a modulo 3 of people in the room.

Removing 1 person is equivalent to $-1 \pmod 3$ while adding 2 people is equivalent to $2 \pmod 3$. We find that the modulo of 3 stays the same (invariant) despite which process is used—this is because

$$-1 \equiv 2 \pmod 3.$$

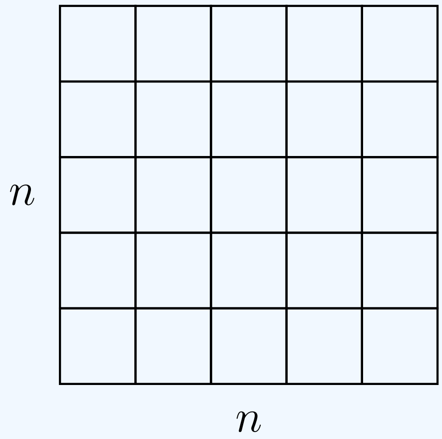
Therefore, the number of people in the room after the initial change in modulo 3 will stay the same as the final number of people modulo 3.

Initially, the number of people in the room is 0, so the modulo 3 starts at 0. The modulo of 3 must stay the same after 3000 minutes, since 3000 is divisible by 3, and this process repeated 3000 times would make the initial $0 \pmod 3$ stay at 0, regardless of any of the -1 or +2 steps. Therefore, since 298 is not congruent to 0 modulo 3 (as $298 \equiv 2 \pmod 3$), this is impossible. †

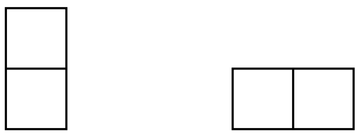
Let's now explore another example in which invariants are applied with the idea of parity to solve the problem.

Example 3.4.3

Take a $n \times n$ board, where $n \in \mathbb{Z}^+, n = 1 \pmod 2$. Is it possible to fit this entire board with either a 2×2 block horizontally or vertically? The board is shown below.



Solution. Note that you must fit the board using these two blocks:

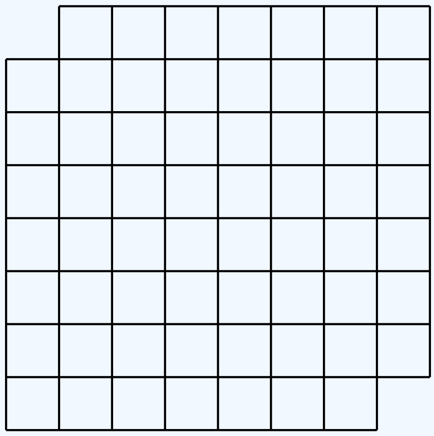


There are several ways to establish this, but one simple way is to look at the parity. We understand that since n is odd, an odd times an odd yields an odd amount of total pieces. Since the pieces we are given are both of even size, it is impossible to multiply an even number by some integer to obtain an odd integer. We see an invariant in which after each piece of two blocks we place, the parity of the number of blocks left to cover does not change. †

Example 3.4.4

Given a 8×8 board below with two of its corners removed, is it possible to fit the entire board with either of two blocks: a vertical or horizontally stacked 2-squares? The blocks are again shown below:

Here is the board:



Solution. This problem is essentially a chess board without its corners. You may attempt to place them one by one, but we can use monovariants instead. What is the common occurrence? Well, let's draw a colored chess board to represent this (Figure 3.4.1):

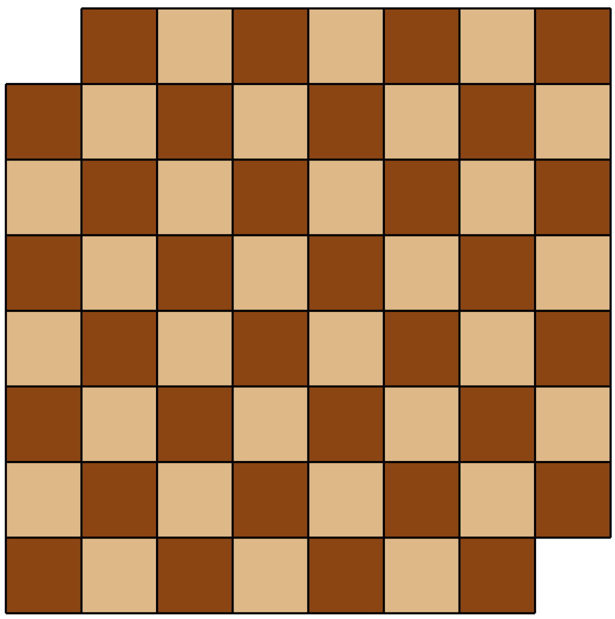


Figure 3.4.1: Chess Board without Two Opposite Corners

Notice that when you place down any of the 2-blocks, you will cover one dark brown spot and one light brown spot. We start off with $8 \times 8 = 64$ spots, and there are initially $\frac{64}{2} = 32$ light brown and 32 dark brown squares. The two corners were light brown squares, and therefore, taking away two light squares ends up with 32 dark brown squares and 30 light brown squares.

Placing one of the two-square pieces down, we cover one dark brown and one light brown square, which ends up being 31 dark brown and 29 light brown left. Placing another piece, there are 30 dark and 28 light. The pattern here is that the number of dark squares minus the number of light squares is always going to be 2, because each piece always covers one dark and one light.

Therefore, we see an invariant, a thing that does not change, between the number of dark and light squares no matter how many pieces we place down. Therefore, there will always be an imbalance and the difference will not be changed, meaning that it is impossible to cover this board with a piece covering two squares horizontally or vertically.

†

Another useful tool in tracking these changes are by identifying monovariants in a process.

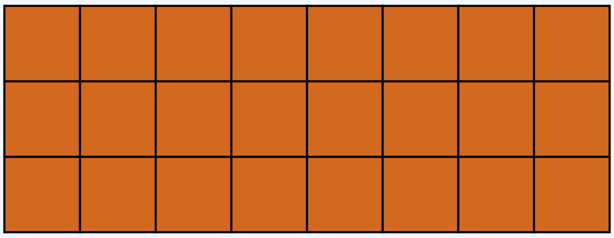
Definition 3.4.5 (Monovariants)

Monovariants are quantities that only increase or decrease, which should be a manageable quantity that can be used to deduce a pattern in a process.

This definition is similar to a definition of a monotonic function, which is a sequence that is always either not increasing or not decreasing. This will also be mentioned in chapter 6.

Example 3.4.6

Steven and James are playing a game where they are splitting a 3×8 chocolate bar. They take turns splitting until you cannot split the chocolate bar any more (i.e., there is only one square of chocolate left). The person who cannot split any more pieces (any pieces that are able to be broken) loses this game. If James starts by splitting this chocolate bar, then will James win or lose? This chocolate bar is shown below.



Solution. We start by inspecting the number of chocolate pieces after each turn. After splitting the chocolate bar once, there would be 2 pieces. After splitting one of those split pieces, there are 3 pieces. Therefore, a monovariant exists in which after each split, there is one more piece. Splitting one time would then entail one extra piece, and so the n-th split would have $n+1$ pieces (where we started with 1 piece).

Thus, since the last possible split creates as many pieces as the number of chocolate squares that exists in the initial piece ($3 \times 8 = 24$), the player who takes the 23rd turn obtains $23 + 1 = 24$ pieces and will go on to win. This means that the other player who takes the 24th turn will lose. Since there are only two players, they share the same parity turn for every turn.

Since James starts with turn 1 and Steven takes turn 2 after him, James has odd turns and Steven has only even turns. Therefore, the player who takes the 23rd turn is James, who will win this game as Steven will be unable to split the chocolate pieces any further after that turn (no piece is breakable as they are singletons). †

Example 3.4.7

Say the numbers $1, 2, \ldots, 100$ are written on a wall. A machine takes multiple steps, where at every step, two of those numbers a, b are erased, and the number $a + b - 1$ is added. After 99 of those steps, what number remains?

Solution. Instead of raw computing, let's take a look at a mini example of $1, 2, 3, 4, 5$ first to see if the order matters. Let's start with $a = 1, b = 4$ and so we delete them and add in the number $1 + 4 - 1 = 4$:

$$2, 3, 5, 4.$$

Next, we choose $a = 3, b = 5$, and so they are removed and we add in the number $3 + 5 - 1 = 7$:

$$2, 4, 7.$$

Let $a = 2, b = 7$ this time, and so we remove those and add $2 + 7 - 1 = 8$:

$$4, 8.$$

Finally, we remove those two numbers and add $4 + 8 - 1 = 11$, obtaining 11 as our final answer. Doing this another way, we start with $a = 5, b = 3$, which becomes

$$1, 2, 4, 7.$$

Then, let $a = 1, b = 4$:

$$2, 4, 7.$$

Next, let $a = 2, b = 4$:

$$7, 5.$$

Finally, this becomes 11 again. It seems like, without loss of generality, the order doesn't matter, and in fact, it doesn't. Why? Let's look at this process from another angle, by finding a monovariant. Well, for each step in the process, we see that one is removed from the sum of a, b. Then, it might be clear that after 99 steps, 99 will be subtracted from the total sum of the original arithmetic sequence. This is because the original sum is always preserved and the only thing that is decreasing is 1 per step. Therefore, it is feasible that this is actually a simple problem and the integers from the sequence chosen for a, b does not matter, as they are being summed either way and the only thing that is changing is the subtraction of 1 per step, which is a consistent pattern. Thus, answer is the sum of the original sequence subtracted by 99. This sum is given by the arithmetic sum formula:

$$\begin{aligned} 1 + 2 + \cdots + 100 &= \frac{100(1 + 100)}{2} \\ &= 50(101) \\ &= 5050. \end{aligned}$$

Finally, we subtract by 99 to obtain our final answer:

$$5050 - 99 = \boxed{4951}.$$

†

Exercises 3.4

Exercise 3.4.1

A large estate has 144 chambers. A total of 2020 guests arrive for a gathering at the estate, and they engage in a game: every minute, one of the guests switches from their current chamber to another chamber that holds at least as many guests as the one they were previously in. The event concludes once all the guests have gathered in the same chamber. Prove that this gathering will eventually finish in a finite amount of time.

Exercise 3.4.2

Consider a set of $n \geq 3$ non-zero numbers, $\{1, 2, \ldots, n\}$, written on a chalkboard. Each minute, you select two numbers, a and b, erase them, and replace them with the numbers $a + \frac{b}{2}$ and $b - \frac{a}{2}$. Can you return to the original set of n numbers after performing this process a finite number of times?

Exercise 3.4.3

You are given a sequence of three numbers. The allowed transformation on the sequence is as follows: select two numbers, say a and b, and replace them with $\frac{(a+b)}{\sqrt{2}}$ and $\frac{(a-b)}{\sqrt{2}}$. Starting from the triplet $(2, \sqrt{2}, \frac{1}{\sqrt{2}})$, is it possible to reach the triplet $(1, 2, 1 + \sqrt{2})$ through a series of such operations?

Exercise 3.4.4

On a whiteboard, six zeros and five ones are initially written. At each step, two numbers are erased, and the following rules are applied:

- If the two numbers are the same, write 1 on the board.
- If the two numbers differ, write 0 on the board.

After performing this process for 10 steps, only one number remains. Determine what this number is.

Exercise 3.4.5

You have a stack of $2n+1$ cards. You can shuffle the stack using these two operations:

1. Cut: Remove any number of cards from the top of the stack and place them at the bottom.
2. Perfect riffle shuffle: Take the top n cards and interleave them with the remaining $n+1$ cards.

Show that, regardless of how many times you perform these operations, the cards can be arranged in at most $2n \cdot (2n+1)$ distinct configurations.

Exercise 3.4.6

In a 10×10 grid, nine unit squares are initially infected. At each step, any unit square that has exactly two infected neighbors becomes infected. Determine whether it is possible for the infection to spread to all the unit squares in the grid.

Exercise 3.4.7

Consider a deck of n cards numbered $1, 2, \ldots, n$. The deck is shuffled randomly. Each step involves reversing the order of the top k cards if the top card is k. For instance, if the top card is 4, the deck

$$\underline{4 \quad 5 \quad 2 \quad 3} \quad 1 \quad 6$$

transforms to

$$\underline{3 \quad 2 \quad 5 \quad 4} \quad 1 \quad 6$$

after one step. Prove that no matter the initial arrangement of the cards, card 1 will eventually end up at the top of the deck.

Exercise 3.4.8

The sum of the digits of 44444444 in decimal notation is equal to n. Let m denote the sum of n's digits. If both n and m are written in decimal notation, what is the sum of m's digits?

Exercise 3.4.9

Show that a number is congruent to the sum of its decimal digits modulo 9.

Exercise 3.4.10

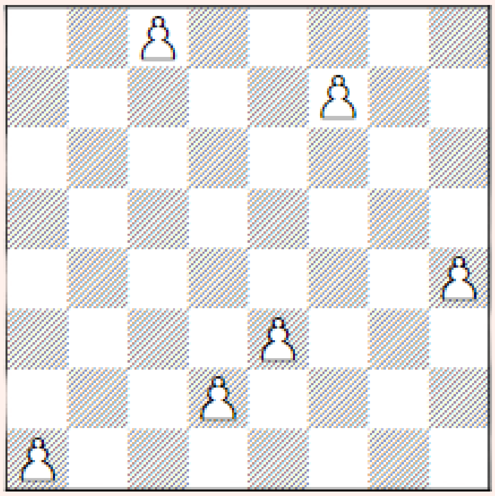
Let n be an odd integer, and consider the sequence $(a_1, a_2, \ldots, a_n)$ as a permutation of the numbers $(1, 2, \ldots, n)$. Prove that the product

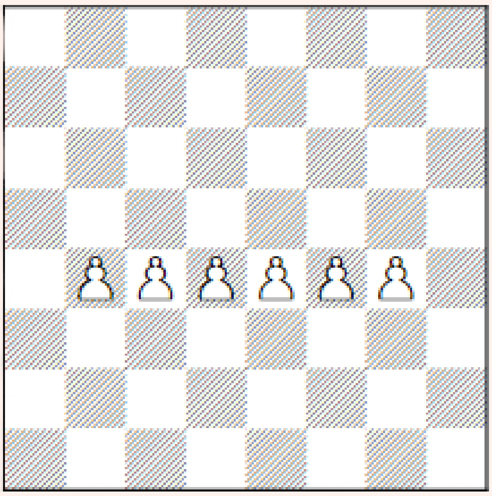
$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is always even.

Exercise 3.4.11

Given six pawns on an 8×8 chessboard, which is positioned as shown in the image below on the left, determine if it is possible for these pawns to eventually end up in the arrangement shown on the right, where each pawn can only move to some horizontally or vertically adjacent square, and multiple pawns can occupy the same square. (Ignore standard chess rules.)





3.5 Pigeonhole Principle

We previously looked at injections between finite sets. Take sets A, B. If $|A| > |B|$, then there cannot exist an injective function from set A to set B. Why? Think of the core definition of an injective function (Definition 3.3.1)—this extends to one of the simple (and intuitive) yet fundamental and useful theorems of combinatorics called the Pigeonhole Principle, also known as Dirichlet box principle (the old definition).

This principle applies in many areas and kind of problems, which you may have previously used, but may be quite useful to refer to in order to convey this point directly and efficiently, and lead to interesting applications in many fields such as computer science, number theory, graph theory, and more.

Let's take a look at 3 specific theorems of the Pigeonhole Principle, in which we will then transition to some examples to display an application of the theorems.

Theorem 3.5.1 (Pigeonhole Principle Part 1)
If $n+1$ pigeons fly into n pigeonholes then one (or more) of the holes must contain (at least) two pigeons.

Think about this theorem and why it is true—it should be quite intuitive. I suggest looking at the "best possible" or the "most extreme" case in which this may be false. We will proceed to prove by contradiction (for any assumptions it is generally a good practice to start by stating it is an assumption for the sake of contradiction).

Proof. Assume for the sake of contradiction that every pigeonhole contains at most one pigeon. Then, even in the case in which every hole contains the greatest amount of pigeons (that is, 1 pigeon per), the total number of pigeons would still be less than the amount of pigeons that flew into the holes:

$$1 \cdot n < n + 1.$$

Additionally, the general claim for this case would then conclude to

$$n + 1 \geq n \cdot 1.$$

Thus, this simplifies to

$$1 \leq 0,$$

which is a contradiction. □

Example 3.5.2
Let

$$X = [x] = \{1, 2, \ldots, x\}$$

and

$$Y = [y] = \{1, 2, \ldots, y\}.$$

Assume that $x > y$. Find the number of functions $f : X \to Y$ in which the mapping is one-to-one.

Solution. By the definition of a one-to-one function (Definition 3.3.1),

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

Therefore, if we assume for the sake of contradiction, then for every object in the set Y, there can be at most one item from X that is mapped onto it. In other words, if this function is injective, then the size of Range(f) has to be less than or equal to the cardinality of Y.

By the Pigeonhole Principle Part 1, if $|X|> |Y|$, and there is a mapping from items from X to Y, then at least one of the "bins" in Y will contain more than one element from the domain. $\therefore$ it is obvious that the range is greater than the maximum range if this were an injective function (where the range is equal to x).

This means that this case in which we assume injectivity,

$$x \leq y,$$

which is a clear contradiction to the initial statement

$$x > y.$$

†

Example 3.5.3

Building off of the last example, if

$$|X| \geq |Y| = 2,$$

then how many functions are onto?

Solution. We solve by complementary counting. There are only two cases in which this function is not onto: if

$$Y = [2] = \{1, 2\},$$

and if X has more than 2 items, then the only two cases in which one of the two items in Y do not receive a mapping onto of at least one item in X are if every item in the domain is mapped onto either the first element of Y or to the second element. Therefore, we calculate the total number of possible mappings to be

$$2^x,$$

since each element of X has two possible elements to map onto. Therefore, the total number of onto functions is just

$$2^x - 2 = \boxed{2(2^{x-1} - 1)}.$$

†

Theorem 3.5.4 (Pigeonhole Principle Part 2)
If $n-1$ pigeons fly into n pigeonholes, then some pigeonhole must be empty (with no pigeons, or in other words, "pigeonless").

The proof of this can be constructed using a similar proof to that of Pigeonhole Principle Part 1. For that reason, this will be omitted, but feel free to attempt yourself.

Theorem 3.5.5 (Pigeonhole Principle Part 3)
If $kn+1$ pigeons fly into n pigeonholes, then at least one hole will contain $k+1$ or more pigeons.

Proof. Assume for the sake of contradiction that each hole contains k or less pigeons. There are two extreme cases: (1) all pigeons fly into one hole, which (without loss of generality) contradicts this case and (2) all pigeons fly as evenly as possible into each hole. If case (2) leads to a contradiction, then the entire problem will be contradicted, because any other case below the extreme will guarantee a less-even distribution where at least one pigeonhole will contain more than any of the count of pigeons in any hole of the extreme case (2). In case (2), the most even distribution falls if kn of the $kn+1$ pigeons fly into the n holes such that each hole contains k pigeons. This is a flat distribution with all holes containing the same number of pigeons, which does agree with the even-ness extreme. However, the last pigeon must fly into one of these. Therefore, even in the extreme case, one pigeonhole has $k+1$ pigeons, which is a contradiction. □

Note that when $k=1$, we obtain Pigeonhole Principle Part 1. This is a general form that can be useful in many places, such as problems that can be solved through modulo. Take a look at such an example below, which is one common form of the Pigeonhole Principle.

Example 3.5.6
Think of a list of n numbers. Within that set of numbers, there must either be a number that is divisible by n or two whose difference is.

Solution. Note that there cannot be duplicates, as there will be a difference of 0, which is indeed divisible by n. Additionally, there cannot be any multiples of n since they are divisible by n. Finally, there cannot be any two numbers with the same modulo of n, otherwise their difference would be divisible by n. Therefore, we hold that there must be n different numbers in terms of modulo n. However, there cannot be any multiples of n, and therefore, there is bound to be at least one number that is a multiple of n if we want different modulos. There can only be a max of $n-1$ integers that can span the following modulos:

$$1 \pmod{n}, 2 \pmod{n}, \ldots, n-1 \pmod{n}.$$

The one left over integer if we want to write n integers would have to either be divisible by n or repeat a modulo, which would entail that the difference is divisible by n. †

Another way to define the Pigeonhole Principle is the following:
Given a function $f : X \to Y$, where X and Y are finite sets:

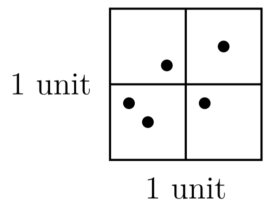
- If $|X| > |Y|$, then f is not one-to-one.
- If $|X| = |Y|$ and f is onto, then f is one-to-one.
- If $|X| = |Y|$ and f is one-to-one, then f is onto.

There are so many examples that come with the Pigeonhole Principle, which will be provided in the exercise section. Feel free to explore those, but we'll go over one more example before reaching the end of this topic.

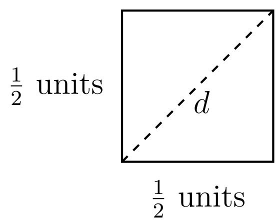
Example 3.5.7

Suppose 5 points are chosen in a unit square. Prove that some points must lie within $\frac{\sqrt{2}}{2}$ units of each other.

Solution. This solution is relatively simple. By the Pigeonhole Principle Part 1, if we split the unit square into four equal sized square quadrants, then one quadrant must contain at least two of the points, including the borders:



One quadrant is shown below, with maximum distance being the diagonal d, which is a fundamental property of either of the right triangles created by the diagonal (that the hypotenuse is the longest side):



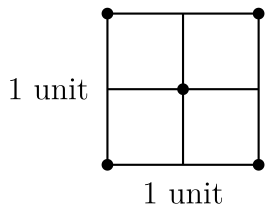
By the Pythagorean Theorem, this distance is equal to

$$\begin{aligned} d &= \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} \\ &= \sqrt{(\frac{1}{4}) + (\frac{1}{4})} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{2}{4}} \\
&= \sqrt{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}.
\end{aligned}
$$

Therefore, the distance between some two points in the same quadrant must be less than or equal to $\frac{\sqrt{2}}{2}$.

Note that the extreme "evenly spread" case is when four points are in the corners and one point is in the center of all the quadrants:



In this case, the middle point will still have a distance of $\frac{\sqrt{2}}{2}$ to the other points, while the other points have a larger distance between them. †

Exercises 3.5

Exercise 3.5.1

A box contains 10 green balls, 10 yellow balls, and 10 black balls. Find the minimum number of balls you will need to draw randomly to ensure that you have at least four balls of the same color.

Exercise 3.5.2

In a mathematics department of the University of Alex's World of Math, study groups can be formed with either 10 calculus students, 8 algebra students, 6 statistics students, or 4 geometry students. What is the smallest number of randomly chosen students required to guarantee that at least one study group will be formed?

Exercise 3.5.3

A group of 13 people are sitting together in a room. Prove that at least two individuals must share the same birth month. (Let there be only 12 months in a year.)

Exercise 3.5.4

11 friends are each having lunch with 12 others. Show that at least one friend must have shared lunch with at least two others.

Exercise 3.5.5

A box contains 10 pairs of red gloves and 10 pairs of blue gloves, all mixed up. Prove that if you randomly pull out three gloves, you are guaranteed to have at least one matching pair, where the gloves are of the same color.

Exercise 3.5.6

Prove that within any group of six people, there must be at least three who either all know each other or none of them know each other.

Exercise 3.5.7

Given a 6×6 grid fully covered with 1×2 dominoes, prove that it is possible to cut the grid along the lines without slicing any domino in half.

Exercise 3.5.8

A coastline spans 50 miles. If 12 governors each claim 10-mile sections of land, prove that there must be a point on the coastline that is contested by at least three governors.

Exercise 3.5.9

Consider a sequence of n numbers. Prove that there exists a consecutive subsequence whose sum is divisible by n. In other words, given any sequence $a_1, a_2, \ldots, a_n$, show that there are indices k and l such that the sum $a_k + a_{k+1} + \cdots + a_l$ is divisible by n.

Exercise 3.5.10

For any natural number n, prove the that there exists a number composed with only the digits 0 and 1 in its binary representation that is divisible by n.

Exercise 3.5.11

Prove that any selection of 10 numbers from the set

$$[19] = \{1, 2, \ldots, 19\}$$

will always contain a pair that sums to 19. After studying Chapter 5, I encourage you to revisit this question to solve using an argument by partitions.

Exercise 3.5.12

Say you select $n + 1$ numbers from the set

$$\{1, 2, \ldots, 2n\}.$$

Show that there will always be two numbers a and b such that $a \mid b$ ("a divides b"). Note that it is possible to choose exactly n numbers from $\{1, 2, \ldots, 2n\}$ such that no pair satisfies this condition.

Exercise 3.5.13

A field is divided into small plots, where each plot is planted with one of three types of crops: wheat, barley, or corn. Show that there is a rectangular section of the field where all four corners are planted with the same crop.

Exercise 3.5.14

The government of AWM contains 51 governors. Say this group of governors is divided into n committees where each governor belongs to exactly one committee, and that each governor likes exactly three other governor (the other governor doesn't necessarily have to like them back). What is the smallest number of committees n in which it is always possible to form and arrange the committees such that no governor hates another governor on the committee they belong to?

Exercise 3.5.15

Given 100 people with randomly selected heights (in centimeters), prove that one can select 15 people such that the difference between the heights of any two selected individuals is divisible by 7.

Exercise 3.5.16

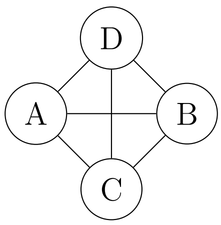
You have seven planks of wood, where each is between 1 and 10 feet in length. Prove that it is possible to choose three of these planks whose lengths can form the sides of a triangle.

3.6 Handshake Lemma

Example 3.6.1

Say you are at a party with four people. Everyone shakes everyone else's hands once. How many handshakes happened?

Solution. The answer is just $3 \times 4 = 12$...right? No, some handshakes are the same, and therefore, this cannot be concluded. Take a look at the figure below:



As you can see, everyone shaked 3 hands. The solution to this problem lies within the fact that each handshake occurs between two people. This means that when we sum up all the handshakes initially, each handshake will be overcounted once, and therefore, the answer is the sum of all the handshakes per person and divided by 2:

$$\frac{4 \times 3}{2} = \boxed{6}.$$

This answer can be confirmed by counting the number of edges that exist in the picture.
†

What we did there was called the Handshake Lemma, which is a fundamental concept for proving many graph theory theorems (these are not covered this in this book) and many other combinatorial problems that involve similar situations that can be dissected with vertices and edges.

Lemma 3.6.2 (The Handshake Lemma)

Given a set of n vertices $v_1, v_2, \ldots, v_n$ with their respective number of edges that they are connected to (called the i-th degrees $d(v_i)$, which form a degree sequence—usually displayed in decreasing order) $(d(v_1), d(v_2), \ldots, d(v_n))$, the number of total edges can be calculated by the following formula:

$$e = \frac{\sum_{i \in \{1,2,\ldots,n\}} d(v_i)}{2},$$

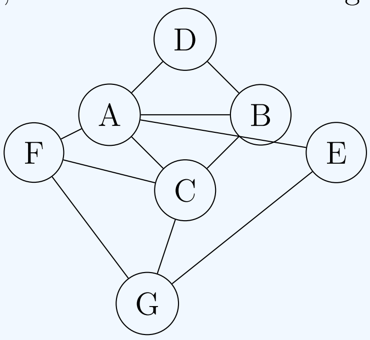
or

$$\sum_{i \in \{1,2,\ldots,n\}} d(v_i) = 2e.$$

The proof is valid with the reasoning given in the previous example problem.

Example 3.6.3

Without explicitly counting the sum of degrees per vertex or "handshakes per person," count that sum of the graph below:



Solution. The Handshake Lemma states that this sum of degrees can be calculated by $2e$. Therefore, we count the number of edges to be 10, meaning our answer is

$$2(10) = \boxed{20}.$$

†

Example 3.6.4

Tommy went to eat dinner. He joined a table of strangers and each of them, including Tommy, exchanged numbers with each other. Here is the degree sequence containing a decreasing order of count of the number of phone numbers that each person at the table got:

$$(5, 5, 5, 4, 3, 3, 2, 2, 2, 1).$$

How many people were at the table, and how many numbers were exchanged in total?

Solution. There are $n = \boxed{10}$ people because there are ten degrees of the ten vertices displayed in that sequence. The number of edges is given by the Handshake Lemma:

$$\begin{aligned} e &= \frac{\sum_{i\in\{1,2,\ldots,n\}} d(v_i)}{2} \\ &= \frac{(5+5+5+4+3+3+2+2+2+1)}{2} \\ &= \frac{32}{2} \\ &= \boxed{16}. \end{aligned}$$

†

Example 3.6.5

At a math meet, several teachers and students shook hands to greet one another. If there were nine people total and each person shook hands 7 times, how many handshakes occurred?

Solution. Following our normal steps, we use $n = 9$ and the sum of every degree would be 9×7 because each vertex (person) has a degree (# of handshakes) of 7. Therefore, the number of handshakes that occurred is equal to:

$$\begin{aligned} e &= \frac{\sum_{i\in\{1,2,\ldots,n\}} d(v_i)}{2} \\ &= \frac{63}{2} \\ &= 31.5. \end{aligned}$$

Therefore, there are 31.5 handsh... Wait—that's not possible! That is correct, there cannot be a non-whole number number of handshakes that occurred. Therefore, this problem cannot have happened, unless the person who recorded it tallied it wrong. †

Proposition 3.6.6

For any given graph, the number of vertices with all odd degrees must not be odd.

Proof. Let there be n vertices each with degree $d(n)$. Then, the number of total edges is equal to, by the Handshake Lemma,

$$\begin{aligned} e &= \frac{\sum_{i\in\{1,2,\ldots,n\}} d(v_i)}{2} \\ &= \frac{n \cdot d(n)}{2}. \end{aligned}$$

If given that all the degrees are odd, then if n is odd, then $n \cdot d(n)$ will also be odd. If the numerator sum is odd, then the number of edges e will be a non-whole number, which is not possible. Thus, the numerator must be even instead of odd, which would cause $n \cdot d(n)$ to be able to be expressed by

$$(2m) \cdot d(n),$$

where m is some whole number where $n = 2m$. The sum would then evaluate to a whole number,

$$e = m \cdot d(n).$$

□

Exercises 3.6

Exercise 3.6.1

In a room with N people, where $N > 3$, at least one person has not shaken hands with everyone else in the room. What is the maximum number of people in the room who could have shaken hands with everyone else?

Exercise 3.6.2

30 people gathered for the AWM high school basketball tryouts. At the tryouts, there are 20 people who all (pairwise) know each other and 10 people who knows no one. If people who know each other shook hands and people who do not know each other fist bumped, how many fist bumps occurred?

Exercise 3.6.3

In a group of 9 people, is it possible that each person is friends with exactly 3 people in the group? 4?

Exercise 3.6.4

Is it possible that a graph with the degree sequence $(7, 7, 5, 4, 3, 2, 1, 0)$ exists? How about $(7, 5, 3, 2, 1)$? If possible, draw a graph that fits this situation.

Exercise 3.6.5

Before Sally's birthday party, two people knew one other person, five people knew two other people, and the rest knew three other people. There were a total of 15 pairs of people who knew each other before the party. How many people that were going to the party knew three other people before the party?

Exercise 3.6.6

Assuming that friendship is symmetric (that is, if a is a friend of b, then b is also a friend of a), is it possible for each person to have 3 friends in a group of 5 people? How about in a group of 6? If possible, draw a graph of this situation.

3.7 Probability Theory

Example 3.7.1

A fair coin is flipped 10 times. Find the probability of obtaining exactly 5 heads in the 10 flips.

There are many ways of solving this problem. One such way is to use combinatorics to find the number of ways to do such a task and then obtaining the probability by dividing by the number of total ways. Note that this is based on the definition of probability (Definition 3.7.2 is shown below).

Definition 3.7.2 (Probability)

The probability, or the chance or likelihood, of event E occuring under some environment is given by

$$P(E) = \frac{\text{number of outcomes in event E}}{\text{number of total outcomes}}.$$

Solution. Following these solution steps, we start by finding the number of ways to obtain exactly 5 heads in 10 flips. This is simply permutating 5 heads into 10 slots, and then dividing by the number of rearrangements possible, which is 5!. One such permutation can be represented below:

$$\underset{\text{slot 1}}{\underline{\text{H}}} \quad \underset{\text{slot 2}}{\underline{\text{H}}} \quad \underset{\text{slot 3}}{\underline{\text{H}}} \quad \underset{\text{slot 4}}{\underline{\text{T}}} \quad \underset{\text{slot 5}}{\underline{\text{T}}} \quad \underset{\text{slot 6}}{\underline{\text{T}}} \quad \underset{\text{slot 7}}{\underline{\text{T}}} \quad \underset{\text{slot 8}}{\underline{\text{H}}} \quad \underset{\text{slot 9}}{\underline{\text{T}}} \quad \underset{\text{slot 10}}{\underline{\text{H}}}$$

In other words, this is just $\binom{10}{5}$ because we choose 5 heads out of 10 flips, and the rest slots will be automatically chosen to be tails. Therefore, we evaluate that there are $\binom{10}{5} = \boxed{252}$ ways of doing this.

Next, the number of total possible permutations is given by the number of options per slot, which is 2 (heads or tails):

$$\underset{\text{slot 1}}{\underline{2}} \quad \underset{\text{slot 2}}{\underline{2}} \quad \underset{\text{slot 3}}{\underline{2}} \quad \underset{\text{slot 4}}{\underline{2}} \quad \underset{\text{slot 5}}{\underline{2}} \quad \underset{\text{slot 6}}{\underline{2}} \quad \underset{\text{slot 7}}{\underline{2}} \quad \underset{\text{slot 8}}{\underline{2}} \quad \underset{\text{slot 9}}{\underline{2}} \quad \underset{\text{slot 10}}{\underline{2}}$$

By the multiplication principle, we obtain $2^{10} = \boxed{1024}$ total outcomes.

Therefore, by Definition 3.7.2, the probability of obtaining exactly 5 heads in 10 flips is equal to

$$\frac{252}{1024} \approx 0.246 \text{ or } \boxed{24.6\%}.$$

†

In probability theory, there are several basic definitions to know.

Definition 3.7.3 (Experiment)
An **experiment** is any situation involving random well-defined outcomes. For example, flipping a coin 5 times is considered an experiment.

Definition 3.7.4 (Sample Space)
A **probability space** contains a set Ω containing all random outcomes called the **sample space**.

Definition 3.7.5 (Event Space)
A probability space also contains a set Σ is called the **event space**. Note that elements $E \in \Sigma$ are also subsets of Σ. A depiction of this can be seen in Figure 3.7.1.

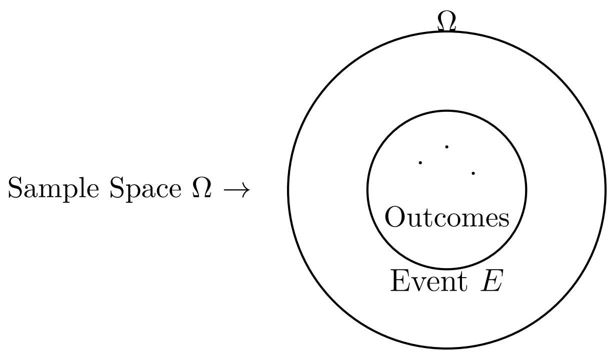


Figure 3.7.1: Event Space

Definition 3.7.6 (Probability Function)
Probability spaces also consist of a probability function $P :\to [0,1]$ such that the following axioms are true:

1. $P(\Omega) = 1$
2. $P(E \cup F) = P(E) + P(F)$ if $\underbrace{E \cup F}_{\text{"mutually exclusive" events}} = \emptyset$. Generally, given any set of pairwise mutually exclusive (disjoint) events
$$\Sigma = \{A_1, A_2, \dots, A_n\}, A_i \cap A_j = \emptyset \text{ for } i < j,$$
then
$$P(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k).$$
Specifically, when $n = \infty$, the collection of infinite events $A_1, A_2, \dots$ represents the axiom of **countable additivity**:
$$P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k).$$

The concept of countable additivity is important when considering infinite unions and many complex problems.

Example 3.7.7
Let $\Omega = \{1, 2, \dots\}$ with probabilities $p_1, p_2, \dots \geq 0$ where $p_1 + p_2 + \cdots = 1$. Additionally, it is true that for any $A \subset \Omega$,
$$P(A) = \sum_{i \in A} p_i.$$
What is the outcome if you toss a fair coin until you land on heads?

Solution. The outcome is simply equal to the number of tosses. The event is flipping a coin until you land on heads. Therefore,
$$p_i = \underbrace{\frac{1}{2} \times \frac{1}{2} \times \dots \times \frac{1}{2}}_{i \text{ times}} = \frac{1}{2^i}.$$

Note that each of these probabilities p_i cannot be equal since it is impossible to sum infinite numbers p_i to equal to 1—infinity times any positive number or probability is equal to infinity. †

Example 3.7.8

Say you flip a coin three times. What is the probability of flipping exactly two heads?

Solution. In this case, $E =$ flipping exactly two heads. We see that

$$E = \{\text{HHT}, \text{THH}, \text{HTH}\}.$$

Each of these three cases occur only once and there are $2^3 = 8$ total outcomes, so the probability of each is $p_i = \frac{1}{8}$.

Each case is disjoint, and therefore, we see that

$$P(E) = \sum_{i \in E} p(i) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \boxed{\frac{3}{8}}.$$

†

Note that probability spaces can have uncountable events that needs to be carefully accounted for. That is,

$$\sum_{x \in E} P(x) = P(E) = 0 + 0 + \dots.$$

There are also a few important results from the axioms to note:

$$(1)\ P(\emptyset) = 0$$

Proof. The general equation in Part 2 of Definition 3.7.6 using all sets to be $\emptyset$ gives the sum of probabilities of $A_1, A_2, \dots, A_n$ where the probability of obtaining an empty set is equal to 0 because there are no ways to obtain an event that does not occur in a sample space. □

$$(2)\ P(E^c) = 1 - P(E)$$

Proof. For the specific case of Part 2 of Definition 3.7.6 where $P(E \cup F) = P(E) + P(F)$ for $E \cup F = \emptyset$, apply $E = E$ and $F = E^c$. By definition, the complement of an event is the rest of the area in an event space that some event E does not cover. Therefore, $P(E \cup F) = 1$, representing the area of the entire probability space. Hence, we obtain $1 = P(E) + P(F)$, or $P(E^c) = 1 - P(E)$. □

(3) $P(F \setminus E) = P(F) - P(E \cap F)$. Note that $B \setminus A$ represents the set subtraction between B and A.

Proof. Any set F is equal to the difference in the events F and some other event E (the set of outcomes that do not appear in event E but appear in F) added to the intersection of events F and E (the outcomes that appear in E and also appears in F). We essentially double counted F, and we can now use the general countable addivity definition (Definition 3.7.6) where $A_1 = P(E \cap F)$ and $A_2 = P(F \setminus E)$, since we know they are disjoint sets. This obtains

$$P((F \setminus E) \cup (E \cap F)) = P(F \setminus E) + P(E \cap F).$$

$P(E \cup F)$ is also equal to $P(F)$ because the union of two disjoint sets is just the sum of the two sets, which in this case we double counted to obtain set F. Therefore,

$$P(F) = P(F \setminus E) + P(E \cap F),$$

or

$$P(F \setminus E) = P(F) - P(E \cap F)$$

. □

(4) If $A \subset B$, $P(B) = P(A) + P(B \setminus A) \geq P(A)$.

Proof. Apply the general countable addivity rule from Definition 3.7.6 where $A_1 = A$ and $A_2 = B \setminus A$. This obtains $P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$, which is a way to double count set B, and therefore $P(A \cup (B \setminus A)) = P(B)$:

$$P(B) = P(A) + P(B \setminus A).$$

Note that since A is a subset of B, the space, or probability, of A will naturally be either smaller or equal to set B (equal set). □

(5) If $E \leq F$, then $P(E) \leq P(F)$.

Proof. As representative of the set size, a probability of an event E, for which its cardinality is less than or equal to the cardinality of event F, is also less than or equal to the probability of event F. □

(6) $P(EUF) = P(E) + P(F) - P(E \cap F)$ — extends to any case by the Principle of Inclusion-Exclusion, where for n events $A_1, A_2, \ldots, A_n$:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) \quad + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k)$$
$$- \cdots + (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right).$$

Proof. Since the set sizes are analogous to probability areas, the proof from normal PIE applies (from Theorem 3.2.5). □

Example 3.7.9

3 children and 4 men sit in a row at random. What is the probability that either all the children or all the men end up sitting next to one another?

Solution. The probability that all children sit next to each other can be thought of as treating the three children as one large object that takes up three seats, and the 4 men sit in the rest of the seats. This is the same as having 4 men and one block of three children being rearranged, which is just

$$5 \times 4 \times 3 \times 2 \times 1 = 5! =$$

.

Since we are looking for ordered arrangements and we treated the children as one unordered block, we multiply by the number of possible rearrangements of the children to find the total number of ways this can be possible. This is just 3!.

Therefore, the total number of ways to arrange 3 children next to one another is equal to $5! \times 3!$.

To find the denominator of the probability, we know that the total number of choices is equal to the number of total permutations, which is $P_7^7 = 7!$. This can be alternatively calculated through the number of choices for each slot. There are 7 people that the first seat can have, and after that one person sits in the seat, there are 6 more for the second, and so on:

$$\underset{\text{slot 1}}{\underline{\quad 7 \quad}} \cdot \underset{\text{slot 2}}{\underline{\quad 6 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 5 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 4 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 3 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 2 \quad}} \cdot \underset{\text{slot 3}}{\underline{\quad 1 \quad}} = 7!\,.$$

Therefore, we find that the probability of children sitting next to one another is equal to $P(C) = \frac{5!3!}{7!}$.

The probability for men can be calculated using a similar manner, where we treat the 4 men as one block, and the other 3 children to fit the rest of the spots. This is the same as the number of ways to rearrange 4 total objects, which is 4!. We then multiply by the number of possible rearrangements in the block of men, which is 4!. Thus, the number of ways to arrange the 4 men and 3 children such that the 4 men are next to each other is equal to $4! \times 4!$.

The total number of permutations is also equal to 7!, so we obtain $P(M) = \frac{4!4!}{7!}$.

We are done...? No, there are cases in which these intersect. Using the probability version of PIE, the number of ways in which either all men or all children are sitting together (or alternatively, $P(C \cup M)$) is equal to the probabilities added together of each case subtracted by the intersection of both:

$$P(C \cup M) = P(C) + P(M) - P(C \cap M).$$

To find $P(C \cap M)$, we can apply the same reasoning with the previous cases: we treat the 3 children as one block AND the 4 men as another distinct block. There would be $2 \times 1 = 2!$ many ways to rearrange these two blocks. To count the ordered arrangements, we multiply by the number of rearrangements within each block, which is $3!\,4!$. Therefore, the number of ways to have children together and all men sitting next to each other is equal to $|C \cap M| = 2!\,3!\,4!$.

The probability can be calculated by dividing by the number of possible permutations, which is again 7!. So, the probability of the intersection is $P(C \cap M) = \frac{2!3!4!}{7!}$.

Finally, we obtain the final answer to be the following:

$$\begin{aligned}
P(C \cup M) &= P(C) + P(M) - P(C \cap M) \\
&= \frac{5!\,3!}{7!} + \frac{4!\,4!}{7!} - \frac{2!\,3!\,4!}{7!} \\
&= \frac{5!\,3! + 4!\,4! - 2!\,3!\,4!}{7!} \\
&= \frac{720 + 576 - 288}{5040} \\
&= \frac{1008}{5040} \\
&= \boxed{\frac{1}{5}}.
\end{aligned}$$

†

Consider a sample space Ω where an outcome w is constant, and the following is true:

$$\exists C \text{ s.t. } P(w) = C \forall w \in \Omega \text{ and } C > 0.$$

This constant definition for every outcome is representative of the same probability across all outcomes w.

Based on axiom 1 from Definition 3.7.6,

$$P(\Omega) = 1.$$

$P(\Omega)$ can be rewritten as the sum of the probabilities of all possible outcomes in Ω:

$$\sum_{w \in \Omega} P(w) = 1.$$

Since each outcome is equal to the constant C for all $w \in \Omega$, we can substitute $P(w) = C$ into the equation:

$$\sum_{w \in \Omega} C = 1.$$

A uniform sum is equal to the constant summand times the number of iterations, just like the area of a rectangle. Thus, substituting the sum with $C \cdot |\Omega|$, we obtain the following, where $|\Omega|$ yields the total number of outcomes in the sample space (by definition, a sample space consists of outcomes and therefore, the size of it is the number of outcomes):

$$C \cdot |\Omega| = 1.$$

Solving for C, we get that

$$\boxed{C = \frac{1}{|\Omega|}}.$$

Next, let event E be a subset of Ω. Then, the probability of E is equal to the sum of all probabilities of outcomes of E, and with a constant outcome probability for each outcome in Ω, the probability of E is the same as:

$$P(E) = \sum_{w \in E} P(w) = \sum_{w \in E} \frac{1}{|\Omega|} = \boxed{\frac{|E|}{|\Omega|}}.$$

This kind of sample space that has these properties is called a uniform probability distribution.

Definition 3.7.10 (Uniform Probability Distribution)

A uniform probability distribution is a special case in which all outcomes w in a finite sample space Ω have the same probability. This applies to cases in which there are no outcomes that are more likely to occur than another, such as a fair coin or fair dice. The probability of each outcome in a uniform probability distribution in which the number of outcomes is equal to $|\Omega|$ is given by

$$P(w) = \frac{1}{|\Omega|}.$$

Consequently, the probability of any event E in the same uniform probability distribution sample space Ω can be calculated by

$$P(E) = \frac{|E|}{|\Omega|}.$$

Example 3.7.11

Suppose there is a class that consists of 8 students. If a team of 3 of those students is to be chosen, what is the probability that all three chosen students are boys, given that there are four boys and four girls?

Solution. This is a uniform probability distribution problem, where each unique team of 3 has an equally likely chance of being formed. Thus, to calculate the event in which all members are boys, we apply the following formula from Definition 3.7.10:

$$P(E) = \frac{|E|}{|\Omega|}.$$

In this case, the number of combinations that a team of all boys can be formed (events) is equal to the number of ways for 4 boys to choose the 3 spots on the team:

$$|E| = \binom{4}{3}.$$

Next, the total number of unique combinations of teams that can be formed (outcomes) is equal to the total number of students choosing the 3 spots, which is just:

$$|\Omega| = \binom{8}{3}.$$

Thus, the probability of choosing a team with all boys is equal to

$$\begin{aligned} P(E) &= \frac{|E|}{|\Omega|}. \\ &= \frac{\binom{4}{3}}{\binom{8}{3}} \\ &= \frac{4}{56} \qquad = \boxed{\frac{1}{14}} \end{aligned}$$

†

Definition 3.7.12 (Conditional Probability)

Conditional probability gives the probability of event E given the event F, which is treated as the new sample space. This also can be thought of as the probability that event E occurs assuming that F already occurred, represented by the diagram below (Figure 3.7.2).

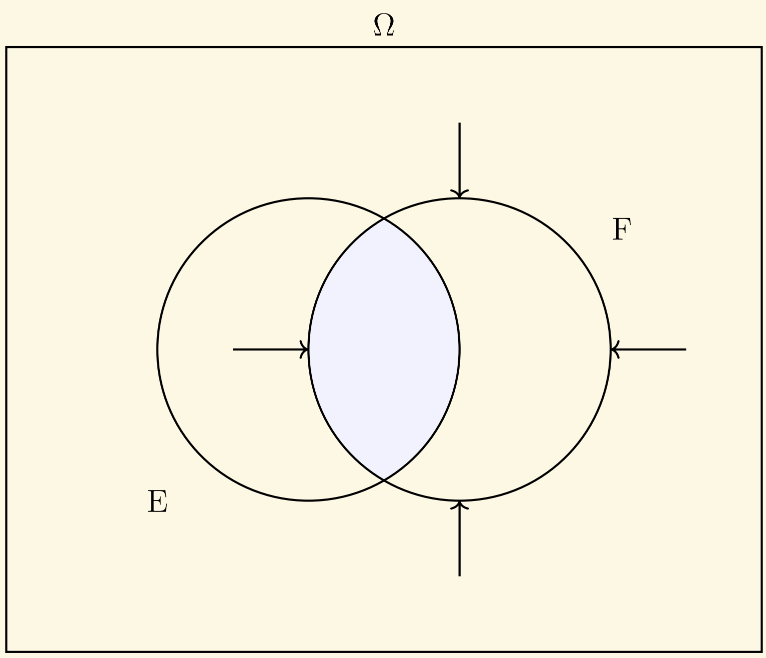


Figure 3.7.2: Conditional Probability of E given F

Theorem 3.7.13 (Conditional Probability Formula)
For events E and F with $P(F) \neq 0$, the conditional probability of E given F is given by:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}. \tag{3.5}$$

Proof. Given F as the new sample space, the only area of outcomes that event E occurs will be $E \cap F$. Therefore, the size of these areas follows the following formula:

$$|(E|F)| = \frac{|E \cap F|}{|F|}.$$

Since the size of events is analogous to the probability or likelihood of those events happening within the sample space, we obtain the probability version of the formula:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

□

Example 3.7.14
Say you have a standard deck of 52 cards. A card is drawn randomly from the deck. What is the probability that the chosen card is a King given that it is the spades suit?

Solution. To find $P(E \cap F)$, where E is the event of drawing a King and F is the event of drawing spades, we count the number of such instances. There is only one of the four possible King cards that are a spades, the King of spades. Since there are 52 cards as the initial sample space, the probability of this occuring is

$$P(E \cap F) = \frac{1}{52}.$$

To find the new specified/given sample space $P(F)$, we count the number of distinct cards that can be drawn from the 52 cards that are spades. There are 13 possibilities (13 cards per suit) and we again divide by 52 total cards to obtain the probability

$$P(F) = \frac{13}{52}.$$

Thus, the probability of picking a King given that the card is a spades suit is

$$\begin{aligned} P(E|F) &= \frac{P(E \cap F)}{P(F)} \\ &= \frac{\frac{1}{52}}{\frac{13}{52}} \end{aligned}$$

$$= \boxed{\frac{1}{13}}.$$

†

Definition 3.7.15 (Independence of Events)
E, F are independent events if

$$P(E|F) = P(E)$$

and

$$P(F|E) = P(F).$$

Theorem 3.7.16 (Intersections of Independent Events)
If events E and F are independent, then

$$P(E \cap F) = P(E) \cdot P(F).$$

Note that this theorem is a specific case of the multiplication principle, which states that if independent events A, B have $|A|$ and $|B$ ways of performing them, respectively, then the number of ways to do both is equal to $|A| \cdot |B|$. In this case, we are specifying the probability version of this theorem, similar to many of the previously discussed axioms such as the one for PIE.

Proposition 3.7.17
The intersection of events E_1, E_2, and E_3 are given by

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 \cap E_2).$$

Proof. The conditional probability of an event E given an event F is defined by Theorem 3.7.13:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \quad \text{where} \quad P(F) \neq 0.$$

By multiplying both sides and through the symmetric property of equality, we can obtain an expression for the probability of the intersection between E and F:

$$P(E \cap F) = P(F) \cdot P(E|F).$$

Now, let's apply this idea to three events E_1, E_2, and E_3. The goal is to find the probability of all three events happening, $P(E_1 \cap E_2 \cap E_3)$.

First, the definition of conditional probability gives an expression for $P(E_1 \cap E_2)$:

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2|E_1).$$

Next, that same definition can be used to form an expression for $P((E_1 \cap E_2) \cap E_3)$:

$$P(E_1 \cap E_2 \cap E_3) = P(E_1 \cap E_2) \cdot P(E_3|E_1 \cap E_2).$$

Finally, substitute $P(E_1 \cap E_2)$ from the equation above with $P(E_1) \cdot P(E_2|E_1)$:

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 \cap E_2).$$

□

The recursive process of this can be seen. For example, for four events, you may treat the intersection of the first three events as the first event in the conditional probability theorem, and the fourth event to be the second event in the theorem. Then, it is evident that this is equal to the probability of the last event given the first three events. This can be recursively extended to n events, $E_1, E_2, \ldots, E_n$, where we obtain the general form of the chain rule for conditional probability.

Corollary 3.7.18 (Chain Rule of Probability)

The chain rule for conditional probability is the result of an iteration of applying the conditional probability formula (Theorem 3.7.13). The intersection of any events $E_1, E_2, \ldots, E_n$ can be described using conditional probabilities:

$$\begin{aligned} P(E_1 \cap E_2 \cap \ldots \cap E_n) =& P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1 \cap E_2) \\ & \cdots P(E_n|E_1 \cap E_2 \cap \ldots \cap E_{n-1}). \end{aligned}$$

Example 3.7.19

Say you deal 3 cards to someone drawing from a standard deck of 52 cards. What is the probability that all three cards all Kings, without replacement?

Solution. There are many ways to evaluate this probability:

(1) Conditional thinking:

Let E_1 represent drawing the first King with no given conditions. Let E_2 represent drawing the second King given that the first King is drawn. Let E_3 represent drawing the third King given that the first two Kings are drawn.

These events are dependent due to there being no replacement. Therefore, by Theorem 3.7.18, we obtain the following expression for the intersection of these three events, which gives the number of ways to perform all of them:

$$P(\text{Three Kings}) = P(\text{First King}) \times P(\text{Second King}|\text{First King})$$

$$\times P(\text{Third King}|\text{First Two Kings}),$$

or

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2|E_1) \times P(E_3|E_1 \cap E_2)$$

Now we can do mini casework to find each of these expressions. For $P(E_1)$, there are 4 possible Kings to be drawn from the 52 standard deck of cards with no restrictions. So, the probability of E_1 is equal to

$$P(E_1) = \frac{4}{52} = \frac{1}{13}.$$

Next, $P(E_2|E_1)$ is the probability of drawing a second King now that the first King is already drawn from the deck of 52 cards. After the first King is drawn, there will be 3 Kings left and 51 cards. This is

$$P(E_2|E_1) = \frac{3}{51} = \frac{1}{17}.$$

Finally, $P(E_3|E_1 \cap E_2)$ is the probability of drawing a third King given that two Kings have already been drawn, leaving 50 cards left and 2 Kings able to be drawn. This is equal to

$$P(E_3|E_1 \cap E_2) = \frac{2}{50} = \frac{1}{25}.$$

Plugging back into the original equation will help us reach our answer.

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P(E_1) \times P(E_2|E_1) \times P(E_3|E_1 \cap E_2) \\ &= \frac{1}{13} \times \frac{1}{17} \times \frac{1}{25} \\ &= \frac{1}{13 \times 17 \times 25} \\ &= \boxed{\frac{1}{5525}}. \end{aligned}$$

(2) Subsets of Size 3

The second way of answering this problem is through counting subsets of size 3 from the set of 52 cards. We want to find the number of ways to choose 3 Kings and divide by the total number of ways to choose 3 cards. There are four Kings that we can choose into our 3 spots. Therefore, there are

$$\binom{4}{3}$$

possible ways of obtaining a hand of 3 cards with 3 Kings. Next, the denominator of the probability is equal to the total number of outcomes in the sample space, which is the number of ways to choose 3 cards using the 52 cards:

$$\binom{52}{3}.$$

Therefore, we obtain the probability of choosing 3 King cards

$$P(3\text{ Kings}) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{4}{22100} = \boxed{\frac{1}{5525}}. \checkmark$$

There are many other ways to result at this probability. We will not explore them for the purpose of this chapter, but feel free to find different combinatorial methods (and perhaps generating functions after reading chapter 4). †

Exercises 3.7

Exercise 3.7.1

David randomly chooses an integer n from $[1, 1000]$. Compute the probability that n is divisible by neither 8 nor 12.

Exercise 3.7.2

12 people sit randomly around a table, including 3 Norwegians, 4 Swedes, and 5 Finns. Find the probability that at least one of these three groups will end up sitting together.

Exercise 3.7.3

The large company ALEX'S WORLD OF MATH consists of n employees. Each employee must purchase a Christmas gift which is then distributed randomly to one of the employees. What is the probability that someone gets his or her own gift?

Exercise 3.7.4

If there are k people in a school classroom, then what is the probability that at least two people share the same birthday? Assume that each birthday is equally likely and come up with an explicit formula to calculate this for any n days instead of $n = 365$ and any k number of students in the classroom.

Exercise 3.7.5

Suppose a die is rolled 12 times. Compute the probability that one of the numbers occur 6 times and the other two occur three times each.

Exercise 3.7.6

You have 10 pairs of gloves in the drawer. You randomly select 8 gloves. For every i, compute the probability that you get exactly i complete pairs of gloves.

Exercise 3.7.7

A researcher flips a biased coin 10 times. Find the following probabilities:
(a) the probability of flipping heads at least once,
(b) the probability of flipping at least one heads and at least one tails,
(c) the probability of flipping exactly three heads, two tails, and five flips landing on the edge.

Exercise 3.7.8

Six employees, consisting of three pairs of coworkers, are randomly seated around a conference table. What is the probability that no coworker sits next to their pair?

Exercise 3.7.9

A group of 20 scientists contains 7 physicists, 3 chemists, and 10 biologists. A committee of 5 people is chosen randomly from this group. What is the probability that at least one of the three professions is not represented on the committee?

Exercise 3.7.10

A 5-character password is formed by selecting each character at random from the set {A, B, C, D, E, F, G, H, I, J}. What is the probability that no character is repeated more than twice?

Exercise 3.7.11

A research team records the daily temperature in a city for 10 consecutive days.
(a) What is the probability that at least one temperature reading is exactly 25°C for 6 days?
(b) What is the probability that at least one temperature reading is exactly 18°C for 3 days?

Exercise 3.7.12

A raffle ticket consists of two rows, each containing 3 items selected from the set $\{1, 2, \ldots, 50\}$. The winning numbers are determined by drawing 5 random items from the same set. A ticket wins if both rows contain at least two of the winning numbers. Compute the winning probabilities for the four raffle tickets shown below:

$$\text{Ticket 1:}\ \begin{matrix} 7 & 12 & 25 \\ 13 & 18 & 22 \end{matrix}$$

$$\text{Ticket 2:}\ \begin{matrix} 4 & 9 & 33 \\ 4 & 9 & 33 \end{matrix}$$

$$\text{Ticket 3:}\ \begin{matrix} 5 & 11 & 27 \\ 9 & 14 & 29 \end{matrix}$$

$$\text{Ticket 4:}\ \begin{matrix} 8 & 15 & 36 \\ 20 & 25 & 40 \end{matrix}$$

For example, if the items 7, 13, 25, 36, and 42 are drawn, Ticket 1, Ticket 2, and Ticket 4 win, while Ticket 3 loses.

Exercise 3.7.13

A box contains two identical keys, one of which opens a door, and one which does not. You are blindfolded and someone tells you that you picked at least one correct key. What is the probability that both keys are correct?

Exercise 3.7.14

A player in a card game flips a card from a shuffled deck 10 times. Knowing (a) exactly 7 flips resulted in a red card, and (b) at least 7 flips resulted in a red card, what is the probability that the first card flip was red?

Exercise 3.7.15

A jar contains 10 green and 10 blue marbles. You draw 3 marbles (a) without replacement, and (b) with replacement. What is the probability that all three marbles drawn are blue?

Exercise 3.7.16

You have two dice: one fair and one loaded to always roll a 6. You randomly choose one die and roll it twice. Both rolls come up as 6. What is the probability that the die you chose was fair?

3.8 Linearity of Expectation

Example 3.8.1

An algebra class consisting of 6 students take a test with a score ranging from 0 to 100. The class scores were the following: $(80, 97, 88, 95, 85)$. But this test was no ordinary test, it was broadcasted for a competition among a few participants to see whoever can guess the closest score that the sixth person was marked. If you were participating in this competition and having known these 5 students' scores, what score would you guess?

Solution. There are many numbers that you may guess. You may use the median, which would be 88 in this case. But a better guess would be the mean, which captures the average value of the dataset. An average value describes the true number that the dataset is revolving around, allowing you to statistically guess the best number possible in the competition. The raw mean can be calculated by summing up all the numbers and dividing by the number of scores:

$$\begin{aligned}\text{Mean} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ &= \frac{80+97+88+95+85}{5} \\ &= \boxed{89}.\end{aligned}$$

†

In statistics, one of the most common measures of a distribution is the mean. Another name for mean is **expected value**. Just like the name suggests, it is the value that the distribution will be expected to follow mathematically after infinite trials. A common example would be flipping a coin and checking the ratio from heads to tails. The initial ratio may be very high when you flip many heads in a row. But after more and more flips, you will start to notice that the ratio will converge to 1. This is an example of a **random variable**.

Definition 3.8.2 (Random Variable)

A random variable is a variable that assigns a numerical value to outcomes in a random experiment. There are two main types of random variable: discrete and continuous. A discrete random variable takes in discrete, or countable, numbers either finite or infinitely, such the number of heads flipped by a coin, while a continuous random variable takes an infinite number of values within a range, such as the weight of a person.

Theorem 3.8.3 (Law of Large Numbers)
The Law of Large Numbers states that for independent variables, the trial distribution mean $\bar{x}$ will converge to the theoretical mean μ as the trials n goes to infinity, with a probability of 1.

Definition 3.8.4 (Expected Value)
The expected value, otherwise known as mean, of a random variable is the average or the center of the probability distribution, representing the true average that the experiment would converge to if repeated infinitely.

Note that the student score example at the beginning of this section contained no duplicates. Let's take a look at a situation where there are duplicates and how we could manage an explicit formula for the mean.

Example 3.8.5
8 year old Maddie decides to sell lemonade with her brother David. Each day, they log their sales in cups of lemonade. The recorded values after 25 days are as shown:

$$1, 1, 1, 2, 3, 4, 5, 5, 3, 4, 5, 7, 5, 4, 10, 3, 4, 6, 3, 4, 7, 5, 6, 10, 12.$$

What is the expected number of cups of lemonade after 25 days?

Solution. Using a simple average, we add

$$1+1+1+2+3+4+5+5+3+4+5+7+5+4+10+3+4+6+3+4+7+5+6+10+12$$

$$= 120.$$

Then, we divide by 25 to obtain the expected number of lemonades to be

$$\frac{120}{25} = \boxed{4.8}.$$

†

Another way to do this is to use the multiplicities of each value, or in this case, the number of lemonades sold on a day to be the weights and we multiply the weights to each value to obtain the weighted sum. Take a look at Figure 3.8.1 below.

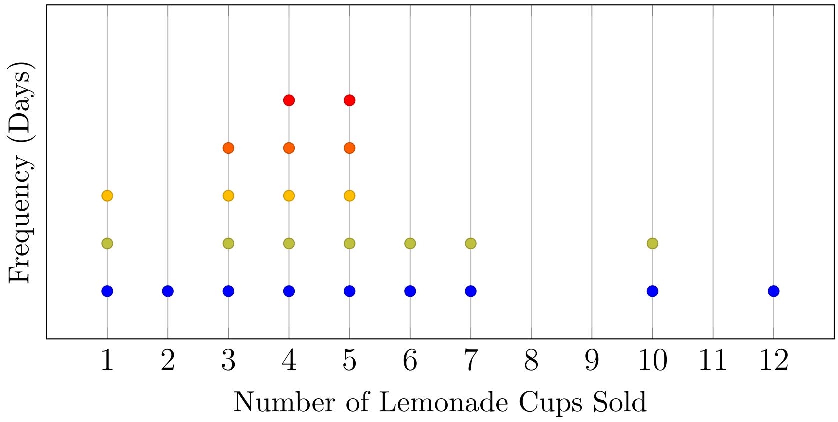


Figure 3.8.1: Distribution of Lemonade Sales

Here, we can calculate the expected value with first the sum

$$3 \cdot 1 + 1 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 5 \cdot 5 + 2 \cdot 6 + 2 \cdot 7 + 2 \cdot 10 + 1 \cdot 12 = 120,$$

and then divide by the number of days, which is 25 to obtain the same answer of 4.8. But we can simplify this even more, by distributing the division of 25 to every single term, obtaining a sum with decimal weights, which all add up to 25:

$$0.12 \cdot 1 + 0.04 \cdot 2 + 0.16 \cdot 3 + 0.20 \cdot 4 + 0.20 \cdot 5 + 0.08 \cdot 6 + 0.08 \cdot 7 + 0.08 \cdot 10 + 0.04 \cdot 12 = 4.8.$$

Since this was an experiment dealing with discrete random variables, this defines an explicit formula used for expected values. Note that weights can be substituted by probability in these random experiments, because probability represents how much each value contributes compared to the whole.

Definition 3.8.6 (Expected Values Formula - Discrete r.v.)

The formula for the expected value $E[X]$ of a discrete random variable X is given by:

$$E[X] = \sum_{x \in \text{Range}(X)} x \cdot P(X = x)$$

where x represent each possible value (outcome) of the random variable and $P(X = x)$, also called the probability mass function, is the probability of the discrete random variable as x.

Example 3.8.7

Imagine that you are waiting for the bus from your home to a park in New York City. How long do you have to wait on average everyday for the bus to come?

Notice that is, in contrast to the lemonade or student score problem, a problem where its values are not discrete—instead, they are continuous. The question is asking for the expected value of a continuous random variable, time. These types of expected values are calculated similar to expected values, as they still multiply by every possible value and assign a probability density function as a weight for each and every value. Instead of discrete values that add up to 1, the probability density function is a non-negative curve in which its area is equal to 1. An example density function can be shown in Figure 3.8.2 below (the curve goes to infinity).

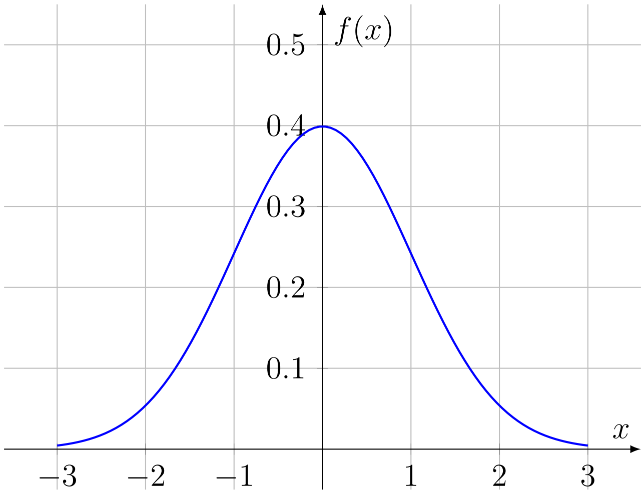


Figure 3.8.2: Probability Density Function

Definition 3.8.8 (Expected Values Formula - Continuous r.v.)
The expected value, denoted by $E[x]$, of a continuous random variable X with the probability density function $f_X(x)$ is given by

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx.$$

If the random variable X takes in values that are restricted ($a \leq X \leq b$), then the expected value is given by

$$E[X] = \int_{a}^{b} x f(x)\, dx.$$

Notice how similar this form of expected value is to the discrete one, but just with the summation switched with an integral, while the probability mass function is replaced by a probability density function. In the bus example, the expected value would be restricted to be non-negative since we are dealing with time, with the lowest possible time to be 0, when the bus is already there and you do not have to wait. The expected value would then take the form

$$E[X] = \int_{0}^{b} x f(x)\, dx.$$

This integral can be depicted by the shaded region (which extends infinitely small to infinity) shown in Figure 3.8.3.

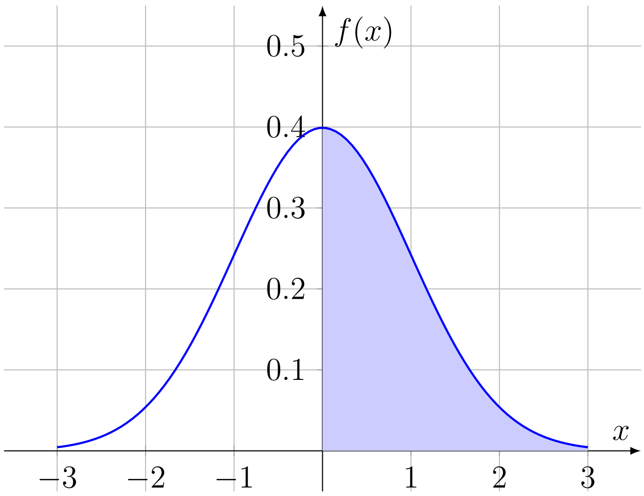


Figure 3.8.3: Probability Density Function of Bus Waiting Time

Let's take at an example of an interesting specific kind of random variable.

Definition 3.8.9 (Indicator Variable)

An indicator variable is a special kind of random variable that gives the contribution per step, which is useful particularly in situations binary values can well represent. For example, the indicator variable for event j is given by:

$$I_j = \begin{cases} 1, & \text{if event } j \text{ occurs,} \\ 0, & \text{if event j does not occur.} \end{cases}$$

Proposition 3.8.10

The expected value of an indicator variable is equal to the probability of the event that the variable represents. Let i_j represent the indicator variable for event j and $P(j)$ be the probability of event j, then the following is true:

$$E[I_j] = P(j).$$

Proof. Let event A be indicated by an indicator variable I_A and has a probability of $P(A)$. Then, by Definition 3.8.6, the expected value of event A is equal to

$$\begin{aligned} E[A] &= 1 \cdot P(X = A) + 0(1 - P(X = A)) \\ &= P(X = A) \end{aligned}$$

$$= P(A).$$

□

The number of successes in a random experiment is often represented by a random variable X_k, such as flipping a heads, in k independent trials. Each trial includes an indicator random variable I_n that represents either success, with probability p, or failure, with probability $q = 1 - p$.

Example 3.8.11

Players A, B flip a weighted coin of probabilities p, q that represent the probabilities of heads and tails respectively, where $p + q = 1$. The rule goes that if a player flips heads, then they win. Otherwise, it is the other player's turn. If player A starts the game, then what is the probability that player A wins?

Solution. We can split this into two cases and create a system of equations using the equations that are modeled by the two scenarios. Let P_i be the indicator variable that player A wins given that it is player A's turn (event i) and P_j be the indicator variable that A wins given that it is player B's turn (event j). Then, it follows that P_i can be modeled using the expected value formula from Definition 3.8.6 for a discrete random variable. Since we are dealing with an indicator variable, the special property of indicator functions that the expected value is equal to the probability (Proposition 3.8.10) can then be applied to find a formula for the probability of P_i.

P_i has two outcomes: heads or tails. The probability of heads (success) is given by $P(P_i = 1) = P(P_j = 0) = p$, while the probability of losing is given by $P(P_i = 0) = P(P_j = 1) = q$, which would lead to P_j. However, since we want to model our success as player A winning as the sample space, the probability of player A winning if the coin flips tails is given by P_j. Therefore, the expected value formula from Definition 3.8.6 writes the following for the expected value of P_i with outcomes x, which can be either the player wins ($P_{\text{win}=1}$) when the coin is heads or P_j for tails (if player B wins, then player A's probability of winning is equal to 0, $P_{\text{lose}} = 0$):

$$\begin{aligned}
E[P_i] &= \sum_{x \in \text{Range}(P_i)} x \cdot P(P_i = x) \\
&= P(j) \cdot P(P_i = 0) + P_{\text{win}} \cdot P(P_i = 1) \\
&= P(j) \cdot q + P_{\text{win}} \cdot p \\
&= P(j) \cdot q + p.
\end{aligned}$$

By Proposition 3.8.10, we hold that the expected value of P_i is equal to the probability of that event of player A winning given that it is player A's turn (i):

$$E[P_i] = P(i),$$

and therefore,

$$P(i) = P(j) \cdot q + p.$$

Next, another equation can be constructed using the same process. Definition 3.8.6 gives the following, with two outcomes of either landing on heads (player B winning, meaning player A loses, $P_j = 0$) or tails (moving to player A's turn, with successes of P_1):

$$\begin{aligned}
E[P_j] &= \sum_{x \in \text{Range}(P_j)} x \cdot P(P_j = x) \\
&= p \cdot P(P_j = 1) + q \cdot P(P_j = 0) \\
&= p \cdot 0 + q \cdot P(i) \\
&= q \cdot P(i).
\end{aligned}$$

By Proposition 3.8.10,

$$E[P_j] = P(j),$$

and consequently

$$P(j) = q \cdot P(i).$$

We obtain a system of two equations, and solving for $P(i)$ will obtain our solution for finding the probability of player A winning:

$$P(i) = P(j) \cdot q + p,$$

$$P(j) = q \cdot P(i).$$

Substituting the second equation into the first, we obtain

$$P(i) = q \cdot P(i) \cdot q + p,$$

$$P(i) - q \cdot P(i) \cdot q = p,$$

$$P(i)(1 - q^2) = p,$$

$$P(i) = \boxed{\frac{p}{(1 - q^2)}}.$$

†

Distributions dealing with indicator variables are known as a **binomial distribution**. Binomial distributions is the sum of multiple independent and equally distributed Bernoulli trials, which are indicator variables with only two possible (random) outcomes of the experiment, being success (1) or failure (0). This was already discussed, but there is an important theorem that will calculate the binomial distribution function, the probability of achieving x successes within n trials, given p probability of success per trial. This calculation is, as the name suggests, connected to another binomial concept—the binomial coefficient and quite similar to the binomial theorem (as you will see below in Theorem 3.8.12, it is the same but just without the summation).

Theorem 3.8.12 (Binomial Distribution Formula)
The probability of achieving exactly m successes (e.g. flipping heads) in k Bernoulli trials, where the probability of each success is represented by probability p and the probability of each failure is represented by probability q, is given by the binomial probability formula:

$$P(X_k = m) = \binom{k}{m} p^m q^{k-m},$$

where $X_k = I_1 + I_2 + \cdots + I_k$.

Proof. Since each of these trials are independent, then the number of ways that m successes can occur within k Bernoulli trials is equal to

$$\binom{k}{m}.$$

For each of those ways to obtain m successes, there is

$$p^m \times q^{k-m}$$

chance to do so: for each success (1) in the sequence, there is p chance and since there are m successes, that is p^m; similarly, for each failure (0) in the sequence of trials X_k, there is a q chance and since there are $k-m$ failures, that is q^{k-m} chance of failure. Since these events are independent, we multiply the probability per outcome by the number of outcomes that can happen to obtain the probability of that event $X_k = m$:

$$P(X_k = m) = \binom{k}{m} p^m q^{k-m}.$$

□

Example 3.8.13
Find the expected number of heads when flipping a weighted coin with probability p of landing heads and q for tails three times. Then, use this expression to find the expected number of heads when flipping three fair coins.

Solution. Definition 3.8.6 gives the following, where $X_3 = I_1 + I_2 + I_3$ is the binomial random variable containing indicator variables for each of the three flips and m is the possible number of successful trials, which ranges from 0 to 3:

$$\begin{aligned} E[X_3] &= \sum_{m=0}^{k} m \cdot P(X_k = m) \\ &= \sum_{m=0}^{3} m \cdot P(X_3 = m) \end{aligned}$$

$$
\begin{aligned}
&= 0 \cdot P(X_3 = 0) + 1 \cdot P(X_3 = 1) + 2 \cdot P(X_3 = 2) + 3 \cdot P(X_3 = 3) \\
&= 1 \cdot P(X_3 = 1) + 2 \cdot P(X_3 = 2) + 3 \cdot P(X_3 = 3).
\end{aligned}
$$

Theorem 3.8.12 provides the values for each $P(X_3 = m)$:

$$
\begin{aligned}
P(X_3 = 1) &= \binom{k}{m} p^m q^{k-m} \\
&= \binom{3}{1} p^1 q^{3-1} \\
&= 3pq^2,
\end{aligned}
$$

$$
\begin{aligned}
P(X_3 = 2) &= \binom{k}{m} p^m q^{k-m} \\
&= \binom{3}{2} p^2 q^{3-2} \\
&= 3p^2 q,
\end{aligned}
$$

$$
\begin{aligned}
P(X_3 = 3) &= \binom{k}{m} p^m q^{k-m} \\
&= \binom{3}{3} p^3 q^{3-3} \\
&= p^3 q^0 \\
&= p^3.
\end{aligned}
$$

Plug these values into the expected value formula from earlier to obtain the expected value, keeping in mind that $1 = p + q$ by definition of success/failure probabilities:

$$
\begin{aligned}
E[X_3] &= 1 \cdot P(X_3 = 1) + 2 \cdot P(X_3 = 2) + 3 \cdot P(X_3 = 3) \\
&= 1 \cdot 3pq^2 + 2 \cdot 3p^2 q + 3 \cdot p^3 \\
&= 3pq^2 + 6p^2 q + 3p^3 \\
&= 3p(q^2 + 2pq + p^2) \\
&= 3p(q + p)^2 \\
&= 3p(1)^2 && \text{(Def of } p, q\text{)} \\
&= 3p.
\end{aligned}
$$

Therefore, the expected number of heads that will be landed within 3 coin flips is equal to $3p$. Flipping three coin flips means that $p = q = 0.5$, and therefore, the expected value

$$\begin{aligned} E[X_3] &= 3p \\ &= 3(0.5) \\ &= \boxed{1.5}. \end{aligned}$$

†

Example 3.8.14

Say you roll two standard dice where the total on two dice $D_2 = X_1 + X_2$ is equal to the sum of the random variables X_1, X_2 of each of the two dice. Find the expected value of D_2.

Solution. Definition 3.8.6 writes the sum over 36 outcomes form of the expected value:

$$E[D_2] = \sum_{w \in Omega} p(w) D_2(w).$$

This can be alternatively summed over the 11 possible values of the total (2,3,...,12):

$$\begin{aligned} E[D_2] &= \sum_x xP(D_2 = x) \\ &= \sum_{x=2}^{12} xP(D_2 = x) \\ &= 2P(D_2 = 2) + 3P(D_2 = 3) + \cdots + 12P(D_2 = 12). \end{aligned}$$

Now, we find the probability of each possible sum (by finding the number of ways to achieve that sum and dividing by the total number of ways to roll the two dice, which is 36):

$$\begin{aligned} P(D_2 = 2) &= \frac{1}{36}, \\ P(D_2 = 3) &= \frac{2}{36}, \\ P(D_2 = 4) &= \frac{3}{36}, \\ P(D_2 = 5) &= \frac{4}{36}, \\ P(D_2 = 6) &= \frac{5}{36}, \\ P(D_2 = 7) &= \frac{6}{36}, \\ P(D_2 = 8) &= \frac{5}{36}, \end{aligned}$$

$$P(D_2 = 9) = \frac{4}{36},$$

$$P(D_2 = 10) = \frac{3}{36},$$

$$P(D_2 = 11) = \frac{2}{36},$$

$$P(D_2 = 12) = \frac{1}{36}.$$

Then, we can use the symmetry:

$$\begin{aligned}
E[D_2] &= 2P(D_2 = 2) + 3P(D_2 = 3) + \cdots + 12P(D_2 = 12) \\
&= \frac{2 \cdot 1 + 3 \cdot 2 + \cdots + 12 \cdot 1}{36} \\
&= \frac{(2+12) \cdot 1 + (3+11) \cdot 2 + (4+10) \cdot 3 + \cdots + (6+8) \cdot 5 + 7 \cdot 6}{36} \\
&= \frac{(14) \cdot 1 + (14) \cdot 2 + (14) \cdot 3 + \cdots + (14) \cdot 5 + 7 \cdot 6}{36} \\
&= \frac{(14) \cdot (1 + 2 + \cdots + 5) + 7 \cdot 6}{36} \\
&= \frac{(14) \cdot (\frac{5(1+5)}{2}) + 7 \cdot 6}{36} \\
&= \frac{(7) \cdot (5 \cdot 6) + 7 \cdot 6}{36} \\
&= \frac{(7 \cdot 6(5+1)}{36} \\
&= \frac{(7 \cdot 6 \cdot 6}{6 \cdot 6} \\
&= \boxed{7}.
\end{aligned}$$

Therefore, the expected value of summing the face-up numbers when rolling two dice is equal to 7. †

Theorem 3.8.15

Let there be random variables x_1 and x_2. Then, the expected value of $x_1 + x_2$ is equal to the sum of the expected value of x_1 and x_2, individually:

$$E[x_1 + x_2] = E[x_1] + E[x_2].$$

Proof. Let $M := x_1 + x_2$, then

$$E[x_1 + x_2] = E[M].$$

Definition 3.8.6,

$$E[M] = \sum_{x \in \mathrm{Range}(\Omega)} M(x) \cdot P(x).$$

It follows that

$$\begin{aligned}
E[x_1 + x_2] &= \sum_{x \in \mathrm{Range}(\Omega)} M(x) \cdot P(x) && \text{(Def. of Expectations)} \\
&= \sum_{x \in \mathrm{Range}(\Omega)} (x_1(x) + x_2(x)) \cdot P(x) && \text{(Def. of } M\text{)} \\
&= \sum_{x \in \mathrm{Range}(\Omega)} x_1(x) \cdot P(x) + \sum_{x \in \mathrm{Range}(\Omega)} x_2(x) \cdot P(x) \\
& && \text{(Property of Summations)} \\
&= E[x_1] + E[x_2].
\end{aligned}$$

□

As we spectate the proof, we see that the theorem of expectations is the result of the linear property of summations. This theorem of expectations allows us to compute difficult expected value problems into parts, similar to what you can do with generating functions that you will learn in the next topic.

Lemma 3.8.16

For any random variable X and constant $c \in \mathbb{R}$, the expected value follows

$$E[cX] = cE[X].$$

Proof. Let $Y := cX$, then

$$E[cX] = E[Y].$$

By the definition of expectation (Definition 3.8.6),

$$E[Y] = \sum_{x \in \mathrm{Range}(\Omega)} Y(x) \cdot P(x).$$

It follows that

$$\begin{aligned}
E[cX] &= \sum_{x \in \mathrm{Range}(\Omega)} Y(x) \cdot P(x) && \text{(Def. of Expectation)} \\
&= \sum_{x \in \mathrm{Range}(\Omega)} (cX(x)) \cdot P(x) && \text{(Def. of } Y\text{)} \\
&= c \sum_{x \in \mathrm{Range}(\Omega)} X(x) \cdot P(x) && \text{(Property of Summations)} \\
&= cE[X].
\end{aligned}$$

□

Note that this is a very similar proof to the other proof, as these are both results from the linearity/properties of summations (being able to multiply a constant in and out as well as split sums). Combining the lemma and theorem, we obtain the theorem called linearity of expectations.

Theorem 3.8.17 (Linearity of Expectations)
Given any random variable X_1, X_2 and constants $c_1, c_2 \in \mathbb{R}$, the linearity of expectations state that the expected value $E[c_1X_1 + c_2X_2]$ may be split like a linear function:

$$E[c_1X_1 + c_2X_2] = c_1E[X_1] + c_2E[X_2].$$

This theorem can extend its linearity to a general definition:

Corollary 3.8.18
Let $X_1, X_2, \dots, X_n$ be random variables on a sample space Ω and let $c_1, c_2, \dots, c_n$ be any constants. Then the expected value of a weighted sum of random variables is equal to

$$\begin{aligned} E[\sum_{k=1}^{n} c_kX_k] &= E[c_1X_1 + c_2X_2 + \cdots + c_nX_n] \\ &= c_1E[X_1] + c_2E[X_2] + \cdots + c_nE[X_n] \\ &= \sum_{k=1}^{n} c_kE[X_k]. \end{aligned}$$

Let's revisit the sum of the rolls of two dice problem using linearity of expectations.

Solution. Since it was given that $D_2 = X_1 + X_2$, in which X_1, X_2 were the individual dice, we can write

$$\begin{aligned} E[D_2] &= E[X_1 + X_2] && \text{(Def. of } D_2\text{)} \\ &= E[X_1] + E[X_2] && \text{(Linearity of Expectations)} \\ &= E[X_1] + E[X_1]. && \text{(Identical Standard Dice)} \\ &= 2E[X_1]. \end{aligned}$$

Definition 3.8.6 gives us the following expression for $E[X_1]$:

$$\begin{aligned} E[X_1] &= \sum_{x\in \text{Range}(X_1)} x \cdot P(X_1 = x) && \text{(Definition 3.8.6)} \\ &= \sum_{x=1}^{6} x \cdot P(X_1 = x) && \text{(Defining Bounds of Dice Rolls)} \\ &= 1 \cdot P(X_1 = 1) + 2 \cdot P(X_1 = 2) + \cdots + 6 \cdot P(X_1 = 6) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}(1+2+3+4+5+6) && \text{(Equal Prob. Per Number on Standard Dice)}\\
&= \frac{1}{6}\frac{6(1+6)}{2} && \text{(Arithmetic Sum Formula)}\\
&= \frac{1}{6}3(7)\\
&= \frac{7}{2}.
\end{aligned}
$$

Therefore, we obtain the same answer as before: the expected value of 2 standard dice rolls is equal to

$$
\begin{aligned}
E[D_2] &= 2E[X_1]\\
&= 2(\frac{7}{2}) && \text{(Substituting } E[X_1] = \tfrac{7}{2}\text{)}\\
&= \boxed{7}.
\end{aligned}
$$

†

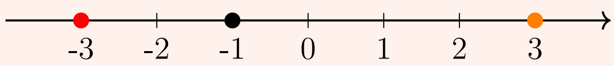
Exercises 3.8

Exercise 3.8.1

A mouse is at position -1 on a horizontal space. He has a $\frac{2}{3}$ probability of moving to the left and a $\frac{1}{3}$ probability to go to the right. If there is cheese at position 3 and a mouse trap at position -3, then what is

1. the probability of the mouse touching the cheese and
2. the average number of steps the mouse will take before reaching either destination.

For reference, this problem can be depicted by the number line below, where the black dot represents the mouse, the cheese is represented by the orange dot, and the red dot represents mouse trap:



Exercise 3.8.2

In a 5×5 garden grid, each plot is randomly filled with either roses or tulips, with equal probability. What is the expected number of 2×2 flowerbeds that are entirely filled with roses?

Exercise 3.8.3

David's sock drawer contains n unique pairs of socks. Each morning, for the next n days, David randomly picks two socks from the drawer to wear. What is the expected number of days he ends up wearing a matching pair?

Exercise 3.8.4

Imagine you are organizing a set of n distinct trophies on a shelf. Each day for the next n days, you randomly select two trophies and place them together on a display stand. What is the expected number of days you have a display with trophies of the same type?

Exercise 3.8.5

A fair die is rolled n times. Define X as the number of segments in the roll sequence where the die shows the same number consecutively. For example, the sequence 3, 3, 1, 1, 2 has three such segments. Calculate the expected value of X.

Exercise 3.8.6

Consider a random arrangement of n different books on a shelf. An inversion is defined as a pair of books (i, j) where $i < j$ and the book at position i is heavier than the book at position j. Let Y be the total number of such inversions. Find the expected value of Y.

Exercise 3.8.7

Tom is looking for 6 specific items in a large box of 30 mixed items. He inspects each item one at a time, in sequence. What is the expected number of items he needs to check before finding all 6 of his desired items?

Exercise 3.8.8

You have five unique digits, 1 through 5, and use them to create two distinct three-digit numbers. For instance, if one number is 241 and the other is 356, determine the expected difference between the two numbers.

Exercise 3.8.9

From a set of numbers ranging from 1 to 90, we randomly select 5 distinct numbers. What is the floor of 10 times the expected value of the fourth largest number in this selection?

Exercise 3.8.10

In a lineup of 20 people consisting of 7 men and 13 women, let S be the count of adjacent pairs where a man (M) is next to a woman (W). For example, in the arrangement

$$MWMWWMWMMWMWMMW,$$

S equals 6. What is the expected value of S across all possible lineups?

Exercise 3.8.11

On a game board of size 8×8, a piece starts at the bottom-left corner and can move to any square in the same row or column every turn, with equal probability. What is the expected number of turns required for the piece to reach the top-right corner?

CHAPTER 4

Generating Functions

4.1 Ordinary Generating Functions

"A generating function is a clothesline on which we hang up a sequence of numbers for display."

Generating functions are an effective tool for breaking down problems into simpler contributing components and then we could just multiply them together to find our desired piece. I will explain this in more detail with an example in just a second.

There are multiple kinds of generating functions such as ordinary generating functions (OGF) and exponential generating functions (EGF)—both are representations that encode the ordered sequence. These generating functions are power series of an ordered sequence.

The general case for OGFs is that $(a_n)_0^\infty = (a_0, a_1, a_2, \dots)$ would have an OGF of

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^3 + \dots &= \sum_{n=0}^{\infty} a_n x^n \\ &= A(x) \\ &= \text{OGF}(a_n). \end{aligned}$$

Assume that for any an sequence that the generating function is denoted by the capital version of it. This sum can often be expressed as a geometric sum, which then we could apply the infinite geometric sum formula

$$\frac{a_0}{1-r}$$

for the ease of simplification.

For example, the OGF of the ordered sequence

$$(1, 1, 1, 1, 1, \dots)$$

can be expressed as

$$1 + x + x^2 + \dots,$$

or the sum of a geometric series using $r = x$ and $a_0 = 1$ as

$$\frac{a_0}{1 - x}.$$

For EGFs, we would often be able to express the sum as an exponential function instead of as a geometric sum so the $(1, 1, 1, 1, \dots)$ sequence would be equal to the general case for EGFs

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

and in this case, an is always equal to 1 so we have

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which is also the Taylor expansion series of ex.

Multivariable generating functions are also another version, feel free to explore those. But this lesson we will introduce with a main focus on OGFs, as they are the easiest to deal with.

It is good to clarify that these functions have a domain of -1 to 1 to make sure that they converge to a finite number. Generating function provides a powerful tool for analyzing (and simplifying/solving) sequences and recurrence relations, as well as understanding many combinatorial problems.

Definition 4.1.1 (Ordinary Generating Functions)

The ordinary generating function (OGF) of a sequence $(a_0, a_1, a_2, \dots)$ is a formal (or "infinite") power series, where the coefficient a_n represents the number of ways to achieve a certain event with n items. The generating function encodes the entire sequence in a manageable (sum) form, which is given by:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots,$$

where x is a formal (temporary) variable, typically bounded by $|x| < 1$ for the purpose of convergence (modeling a finite situation). By definition, the coefficient a_n can be extracted from the generating function as the coefficient of x^n.

Let's look at an example of expressing the sequence as a generating function:

Example 4.1.2

Say we flip a coin one time. Let a_n equal the number of ways to flip n heads after one coin flip. What is the generating function $A(n)$ for this function a_n?

Solution. Using trial for cases, the following is obtained:

$$a_0 = 1(\text{T}) \text{ (One way to flip 0 heads)}$$
$$a_1 = 1(\text{H})$$
$$a_2 = 0$$
$$.$$
$$.$$
$$.$$
$$a_n = 0 \ \forall \text{ ("for all") } n > 2.$$

So, the ordered sequence representation of this problem is given by $(1, 1, 0, 0, 0, \ldots)$. This would imply that there is an OGF(# of heads from a single coin flip) of

$$\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots$$
$$= 1 + x.$$

This would follow a trend for two coin flips instead of 1 that you can just break the coin flips into 2 one flips and multiply the OGFs to find the OGF for 2 coin flips. So in this case,

$$\text{OGF(\# of heads from two coin flips)} = (1 + x)^2.$$

Let me prove this to you. Following the same logic from previously, we can trial each case:

$$a_0 = 1(\text{TT})$$
$$a_1 = 2(\text{HT or TH})$$
$$a_2 = 1(\text{HH})$$
$$a_3 = 0$$
$$.$$
$$.$$
$$.$$
$$a_n = 0 \forall n > 3.$$

$\therefore$ ("therefore") the OGF is

$$\begin{aligned}A(n) &= (a_n)_0^\infty \\ &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0x^0 + a_1x^1 + a_2x^2 + \dots \\ &= 1 + 2x + x^2 \\ &= \boxed{(1+x)^2}.\end{aligned}$$

†

Cool, right?

Now let's look at another example, but instead, let's use generating functions to solve a complex counting problem:

Example 4.1.3

How many ways can you roll a total of 8 using 1 dice from 1 to 2, 1 dice from 1 to 4, and a standard dice?

Solution. For this problem, it is easier to break it into each condition and find the OGF of how much each condition contributes to the problem, then multiply all of them together to find the OGF for the whole problem.

We can start with the 1 to 2 dice. This can only possibly contribute 1 and 2 to the sum:

$$\begin{aligned}\text{OGF(Ways of obtaining n by rolling die 1 once)} &= A(n) \\ &= (0, 1, 1, 0, 0, \dots) \\ &= x + x^2 \\ &= x(1+x).\end{aligned}$$

Repeating for the other two would result in the following expressions:

$$\begin{aligned}\text{OGF(Ways of obtaining n by rolling die 2 once)} &= B(n) \\ &= (0, 1, 1, 1, 1, 0, 0, \dots) \\ &= x + x^2 + x^3 + x^4 \\ &= x(1 + x + x^2 + x^3).\end{aligned}$$

$$\begin{aligned}\text{OGF(Ways of obtaining n by rolling die 3 once)} &= C(n) \\ &= (0, 1, 1, 1, 1, 1, 1, 0, 0, 0, \dots)\end{aligned}$$

$$= x + x^2 + x^3 + x^4 + x^5 + x^6$$
$$= x(1 + x + x^2 + x^3 + x^4 + x^5).$$

The property of OGFs allows us to then use the OGF for each of the three dice to find the OGF for this event that is looking for the intersection of these three independent events.

So,

$$\text{OGF(Total of rolling all 3 dices)} = A(n) \cdot B(n) \cdot C(n),$$

which we can now substitute to solve for the OGF for this problem:

$$\begin{aligned} A(n) \cdot B(n) \cdot C(n) &= x(1+x) \cdot x(1+x+x^2+x^3) \cdot x(1+x+x^2+x^3+x^4+x^5) \\ &= x^3(1+x)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4+x^5). \end{aligned}$$

Here is the power of generating functions: If we just find the coefficient of x^8, it will give us the answer. Why? Well, OGF of a sequence gives a series that satisfies each constraint given in the problem. So because we already filled in how many ways each constraint contributes to the problem by putting them as the coefficients of x^n terms.

When we multiply them together, we can search for the coefficient of x^n after everything is multiplied together to find the number of ways to achieve n with a given set of constraints. What generating functions basically does is hide these answers in the coefficients and use the linearity of generating functions to uncover a desired coefficient as our answer.

So this also means that we are finding the coefficient of x^5 of $(1 + x)(1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4 + x^5)$ since x^3 would have to have multiplied by terms with x^5 to get x^8. Further, the first and second sums can be multiplied to obtain:

$$(1 + 2x + 2x^2 + 2x^3 + x^4)(1 + x + x^2 + x^3 + x^4 + x^5).$$

We also know combinations of terms that multiply to get terms of x^5 are:

1. x^5 in the second sum with 1 in the first sum, obtaining a coefficient of 1
2. x^4 in the second sum with $2x$ in the first sum, obtaining a coefficient of 2
3. x^3 in the second sum with $2x2$ in the first sum, obtaining a coefficient of 2
4. x^2 in the second sum with $2x3$ in the first sum, obtaining a coefficient of 2
5. x in the second sum with $x4$ in the first sum, obtaining a coefficient of 1

$\therefore$ there is a coefficient

$$1 + 2 + 2 + 2 + 1 = \boxed{8}$$

of x^5, and therefore there are 8 ways of obtaining a sum of 8 from rolling the three dice (feel free to find all these ways by hand to confirm this result). †

Now try another example, using geometric sums, which is a core principle for managing these generating functions:

Example 4.1.4

How many ways can you roll a total of 14 using 5 standard dice?

Solution. Note that (1) we can use the Binomial Theorem from Chapter 2:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},\end{aligned}$$

and (2) the finite geometric sum formula is

$$\frac{1-x^n}{1-x},$$

which also supports the one for the infinite, since, by the definition of generating functions,

$$-1 < x < 1.$$

Thus we obtain the formula for an infinite geometric sum,

$$\begin{aligned}\lim_{n\to\infty} \frac{1-x^n}{1-x} &= \frac{1-0}{1-x} \\ &= \frac{1}{1-x}.\end{aligned}$$

Starting with finding the OGF for one dice, we obtain

$$\begin{aligned}\text{OGF(Ways of obtaining n from one dice)} &= (0,1,1,1,1,1,1,0,0,\dots) \\ &= x + x^2 + x^3 + x^4 + x^5 + x^6 \\ &= x(1 + x + x^2 + x^3 + x^4 + x^5).\end{aligned}$$

Then,

$$\text{OGF(Ways of obtaining n from 5 dice)} = x^5(1 + x + x^2 + x^3 + x^4 + x^5)^5$$

We want to obtain the coefficient

$$x^{14-5} = x^9$$

of

$$(1 + x + x^2 + x^3 + x^4 + x^5)^5 = (\frac{1-x^6}{1-x})^5$$

through the finite geometric series formula, since there are $n = 6$ terms. We also apply the binomial theorem to obtain a sum for the geometric series, which makes it easier to find the possible coefficients of x^9:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Thus, we obtain

$$\begin{aligned}
\frac{(1-x^6)^5}{(1-x)^5} &= (1-x^6)^5(1-x)^{-5} \\
&= \sum_{k=0}^{5} \binom{5}{k} (-x^6)^k \sum_{k=0}^{-5} \binom{-5}{k} (-x)^k && \text{(Binomial Theorem)} \\
&= \sum_{k=0}^{5} \binom{5}{k} (-1)^k x^{6k} \sum_{k=0}^{\infty} \binom{-5}{k} (-1)^k x^k.
\end{aligned}$$

So we can consider the cases where x^{6k} and x^k multiplied would make 9. Well, if both k's are integers (keep in mind they are different variables), then the only cases would be the first k denoted by k_1 being 0 and the second k denoted by k_2 being 9 or when $k_1 = 1$ and $k_2 = 3$. Any other case would not work since k_2 makes x^{12}, and

$$k_2 \notin \mathbb{Z}^-$$

so there is no way to decrease the exponent any further.

All we do now is find each case and add them together to find the coefficient of x^9:
$k_1 = 0, k_2 = 9$:

$$\binom{5}{0}(-1)^0 \times \binom{-5}{9}(-1)^9 = (1)(1) \times (-715)(-1) = 715,$$

$k_1 = 1, k_2 = 3$:

$$\binom{5}{1}(-1)^1 \times \binom{-5}{3}(-1)^3 = (5)(-1) \times (-35)(-1) = -175.$$

$\therefore$ the total number of ways of rolling a total of 14 using 5 standard dice is

$$715 + -175 = \boxed{540}$$

ways. †

Notice that this generating functions approach basically provides another way to double count a problem, as we could have obtained the same result by interpreting it as complementary counting of 715 as the total number of ways possible subtracted by 35×5 illegal cases. Pretty interesting, isn't it?

Now try a simple example.

Example 4.1.5

Suppose we are choosing an unordered combination of 3 with repetition allowed from the set $\{A, B, C\}$ such that:

1. The # of A's is even,
2. The # of B's is odd, and
3. The # of C's is ≥ 2.

Solution. We break this problem down into three parts representing each criteria/restriction, and we find their OGF:

$$\begin{aligned}\text{OGF(Ways of following condition 1 of } n \text{ "combinations")} &= (1, 0, 1, 0, \dots) \\ &= 1 + x^2 + x^4 + \dots\end{aligned}$$

$$\begin{aligned}\text{OGF(Ways of following condition 2 of } n \text{ "combinations")} &= (0, 1, 0, 1, \dots) \\ &= x + x^3 + x^5 + \dots\end{aligned}$$

$$\begin{aligned}\text{OGF(Ways of following condition 3 of } n \text{ "combinations")} &= (0, 0, 1, 1, \dots) \\ &= x^2 + x^3 + x^4 + \dots\end{aligned}$$

Therefore, the OGF(Ways of following all 3 conditions of n "combinations") is simply

$$(1 + x^2 + x^4 + \dots)(x + x^3 + x^5 + \dots)(x^2 + x^3 + x^4 + \dots).$$

We could express these as geometric series and then use binomial theorem formulas to obtain the coefficient sums, but this problem can be solved more simply due to the small $n = 3$:

We just want x^3 and looking at the third sum, we know that the only possible routes are if x^3 is multiplied by two constants and if x^2 is multiplied by one x and one constant. For x^3 in the third sum in this case, there is only one available constant in the first sum so it is impossible to obtain a x^3 term with the x^3 in the third sum. Next, x^2 only is able to be matched with x in the second sum and 1 in the first sum to create a x^3 term, of coefficient 1.

$\therefore$ the answer is simply $\boxed{1}$. †

This makes sense, too! We are asked to find unordered combinations, so that means that all orders as long as there are the same objects are considered one combination. To follow condition 3, there must be at least 2 C's. So there must be 2 or 3 C. Now condition 2 says the number of B's must be odd, so the only combination is if there are

1 B and 2 C's. And that would also fit the first criteria of A being even, since 0 is even. So we have one unordered sequence of

$$(\mathrm{B},\mathrm{C},\mathrm{C}) = (\mathrm{C},\mathrm{B},\mathrm{C}) = (\mathrm{C},\mathrm{C},\mathrm{B})$$

that we can obtain by choosing three times from the set $\{\mathrm{A},\mathrm{B},\mathrm{C}\}$.

Congratulations, you've reached the end and unlocked a new, powerful way to approach complex combinatorics problems! Another section in this chapter will focus on exponential generating functions and display useful real-world applications of generating functions for solving recurrence problems in the recurrence chapter. In the exercise section, there will be highly encouraged exercise problems for you to try—it will make sure you grasp a full understanding of this intricate topic. All and all, generating functions are an immensely powerful tool in dealing with and breaking down complex problems.

There are many more generating functions to explore; some of the realms deal with partitions of numbers—partitions of n are the number of ways you can split n into unordered sequences of parts, meaning

$$5 = 1 + 2 + 2 = 2 + 1 + 2$$

are both considered the same partition of 5. The number of partitions of n is mathematically hard to compute and find a clean explicit formula for, which using generating functions helps make it much easier to compute. If this seems very interesting to you, start by trying to find the OGF of the number of partitions for any non-negative integer n. Then, perhaps try to prove that the number of partitions of n into odd parts is equal to the partitions of n into distinct parts. If you do, I assure you, you will be amazed—this will be part of what is presented in the next chapter. Good luck and keep exploring!

Youtube Lectures 4.1

1. [Part 1]Introduction to Generating Functions
2. [Part 2]Introduction to Generating Functions
3. [Part 3]Introduction to Generating Functions

Exercises 4.1

Exercise 4.1.1

Using generating functions, find the number of solutions to $x_1 + x_2 + x_3 + x_4 = 14$, where $0 \le x_1 \le 3$, $2 \le x_2 \le 5$, $0 \le x_3 \le 5$, $4 \le x_4 \le 6$.

Exercise 4.1.2

Derive a generating function for the number of solutions to $x_1 + x_2 + x_3 + x_4 = k$, where $0 \le x_1 \le \infty$, $3 \le x_2 \le \infty$, $2 \le x_3 \le 5$, $1 \le x_4 \le 5$.

Exercise 4.1.3

Given the following equation:

$$3x + 2y + 7z = n,$$

find a generating function for the number of non-negative integer solutions.

Exercise 4.1.4

Suppose there is an infinite supply of red, white, and blue balloons. How many different combinations of 10 balloons are there such that each bunch has at least one balloon of each color and the number of white balloons is even?

Exercise 4.1.5

Prove that every positive integer can be written in exactly one way as a sum of distinct powers of 2 using a generating functions approach.

Exercise 4.1.6

Jacob has two 8-sided dice (numbered from 1 to 8). Joe also has a pair of 8-sided dice that are labeled with positive integers such that for any positive integer n, where $2 \leq n \leq 16$, the probability that the sum of numbers on Jacob's dice will sum to n is equal to the probability that the sum of numbers on Joe's dice is equal to n. Given that one of the numbers labeled on Joe's dice is 11, find the rest of the numbers on Joe's dice.

Exercise 4.1.7

Manipulate the following infinite series expression to yield a closed form:

$$x + 4x^2 + 9x^3 + 16x^4 + \ldots$$

Exercise 4.1.8

Penny rolls 5 standard dice. Find the number of ways to roll a sum of 25 (answer in binomial coefficients).

Exercise 4.1.9

Reason that the number of solutions of the equation $a+b+c+d=n$, where a, b, c, d are odd positive integers, is given by the coefficient of x^n in the following expression:

$$\left(\frac{x}{1-x^2}\right)^4.$$

Exercise 4.1.10

Say n denotes the number of ordered quadruples (x_1, x_2, x_3, x_4) where each x_i is a positive odd integer. Say these quadruples follow this equation:

$$\sum_{i=1}^{4} x_i = 98.$$

Compute $\frac{n}{100}$.

Exercise 4.1.11

An integer is considered balance if the sum of the two leftmost digits of that number is equal to the sum of that of the two rightmost digits. How many of these integers are there that lie between 1000 and 9999 (inclusive)?

Exercise 4.1.12

In a country with an abnormal currency system, six friends share an apartment to live in. This country uses bank notes of denominations of \$1, \$3, \$4, and \$6. One of the friends, Nathan, always carries an unlimited supply of \$3 bills, Michael always has an unlimited supply of \$4 bills, and Steven has that of \$6 bills. On the other hand, Jonathan, Bobby, and Carl have a limited amount of cash: Jonathan has only two \$1 bills, Bobby has three \$1 bills, and Carl has five \$1 bills. How many ways can these six friends pay for an \$n bill?

Exercise 4.1.13

There are ten identical cubes of dimensions 3 ft $\times$ 4 ft $\times$ 6 ft. One of the cubes are placed flat on the floor while the rest of the nine cubes are placed one after another flat on top of the previous cube where the orientation of the cube is chosen at random. If $\frac{m}{n}$ denotes the probability in which these cubes have a height of exactly 41 ft, where m, n are positive and relatively prime integers, compute the value of m.

Exercise 4.1.14

Given that
$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots,$$
find the coefficient of x^n in $\left(\frac{x}{(1-x)^2}\right)^2$ in through two different ways. Then, find a formula for $\sum_{k=1}^{n-1} k(n-k)$.

Exercise 4.1.15

Using the Binomial Theorem, find the coefficient of x^r on both sides of
$$(1+x)^{m+n} = (1+x)^m(1+x)^n.$$
Utilize this to prove Vandermonde's Identity:
$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}.$$

Exercise 4.1.16

Recall the Hockey Stick Identity from Chapter 2. We previously proved it through double counting. Prove the Hockey Stick Identity by comparing coefficients of both sides of the equation, treating each side as its own generating function and holding that they must be equal.
$$(1-x)^{-n} = \sum_{k=0}^{n-1} x^k \binom{n-1}{k}.$$

Exercise 4.1.17

Use generating functions to show that
$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots.$$

Exercise 4.1.18

The polynomial $P(x)$ contains integer coefficients such that the following is true in which $0 < x < 1$:

$$P(x) = \frac{(x^{2310} - 1)^6}{(x^{105} - 1)(x^{77} - 1)(x^{42} - 1)(x^{30} - 1)}.$$

What is the coefficient of x^{2022} in the polynomial $P(x)$?

Exercise 4.1.19

Compute the number of 8-tuples containing non-negative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ such that $0 \le a_k \le 5$ for $k \in \{1, 2, 3, 4\}$, and that the following is true:

$$a_1 + a_2 + a_3 + a_4 + b_1 + b_2 + b_3 + b_4 = 51.$$

Exercise 4.1.20

A library has 100 books available for lending. As you arrive, there are five people ahead of you, numbered from 1 to 5. The kth person in line considers the set that represents the highest number of books remaining through a set of positive integers congruent to 1 modulo 5. If this set was an empty set, then no books are borrowed. Otherwise, one element of this set is chosen and that many books are borrowed. For example, the first person in line picks an element from the set $\{1, 6, \ldots, 96\}$ and borrow that many books. In how many ways can the first five people in line borrow books such that at least 35 books remain in the library?

Exercise 4.1.21

If S denotes the set of all triple positive integers (i, j, k) where $i + j + k = 17$, compute the following:

$$\sum_{(i,j,k)\in S} ijk.$$

Exercise 4.1.22

Given that there exists a positive integer n, let a_n denote the number of permutations π of the numbers $1, 2, \ldots, n$ such that

$$|\pi(i) - i| \leq 2 \forall 1 \leq i \leq n.$$

The generating function of the sequence, $(a_n)_0^\infty$ is given by

$$\frac{1}{1 - 2x - 2x^2 + x^5}.$$

Using this generating function to find a_5.

Exercise 4.1.23

Suppose the following sequence a_n is defined in a way such that

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Show that for each non-negative integer n, there must be some integer m such that

$$a_n + a_{n+1} = a_m.$$

Exercise 4.1.24

Let n be any integer. Prove that using a given set of weights with measures

$$1, 3^1, 3^2, 3^3, \ldots,$$

a n-gram weight (using both pans of a scale) can be weighed—and also show that this can be done in exactly one way.

Exercise 4.1.25

Suppose n and k are non-negative integers. Let $Q(n, k)$ represent the coefficient of x^k in the following expansion: $(1 + x + x^2 + x^3)^n$. Show that

$$Q(n, k) = \sum_{j=0}^{n} \binom{n}{j} \binom{k - j}{n - j}.$$

4.2 Exponential Generating Functions

Every function has a Taylor series, an infinite sum of the function's derivatives (slopes of the function). These series are equal to the function itself, as it captures the shape of the function through infinite derivatives (for functions such as sine, cosine, etc., while polynomials can be defined by a finite number of derivatives that equal the polynomial itself). Taylor series are highly useful for finding other ways to simplify infinite series, or manipulate exponential generating function expressions as we will use throughout this section.

Definition 4.2.1 (Taylor Series)
Taylor series provides every function another representation in the form of an infinite series as a sum of all the function's derivatives (usually focusing at a certain point). The Taylor series of a function $f(x)$ about $x = a$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ calculated at the point $x = a$.

An example is shown below.

Definition 4.2.2 (Taylor Series of the Natural Exponential Function)
The Taylor series for the function $f(x) = e^x$ at the point $x = 0$ is given by:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

This definition example Taylor series will be used very often for exponential generating functions, as will be shown in an upcoming example. A list of popular/commonly used Taylor series can be seen in the open source notes published by USC: https://people.math.sc.edu/girardi/m142/handouts/10sTaylorPolySeries.pdf.

So what are exponential generating functions and how are they different than ordinary generating functions?

Definition 4.2.3 (Exponential Generating Functions)
Exponential generating functions (EGFs) store and encodes information of sequences (e.g. the number of ways to do an event). Instead of only an OGF's $a_n x^n$ in each sum term, $n!$ is divided. EGFs are particularly useful for dealing with events in which order matters (such as permutations and partitions).

Definition 4.2.4 (General Form of Exponential Generating Functions)
Consider the sequence

$$(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \dots).$$

The exponential generating function can then generally be defined as

$$\begin{aligned}\mathrm{EGF}(x) &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \\ &= \frac{a_0 x^0}{0!} + \frac{a_1 x^1}{1!} + \frac{a_2 x^2}{2!} + \dots + \frac{a_n x^n}{n!} + \dots.\end{aligned}$$

Note that EGFs often deal with infinite series containing factorials, and since Taylor series often end up with factorials by definition, Taylor series are commonly used for simplifications of EGFs rather than solely geometric sequence sums.

As previously mentioned, EGFs are useful for counting permutations because they deal with factorials, but specifically of multisets.

Definition 4.2.5 (Multisets)
Multisets are sets that do not follow the traditional rule of distinct objects. Instead, there can be repeated objects in multisets, such as the following sequence of objects:

$$\{a, b, b, b, c, c\}.$$

This expression can otherwise expressed as the following, where the first number is the number of appearances of the item in the multiset:

$$\{a, 3 \cdot b, 2 \cdot c\}.$$

Example 4.2.6
Find the number of permutations of the multiset

$$\{a, a, b\}.$$

Solution. There can be different sized permutations of this, represented by sub-multisets, of size 0, 1, 2, and 3. For size 0, there is one way to permutate the set into nothing (by the Principle of Nothingness). For size 1, there are two ways (since the a's are identical):

$$\{a\}, \{b\}.$$

For size 2, there are three ways:

$$\{a, a\}, \{a, b\}, \{b, a\}.$$

For a 3-size sub-multiset, there are also three ways:

$$\{a, a, b\}, \{a, b, a\}, \{b, a, a\}.$$

Thus, there are

$$1 + 2 + 3 + 3 = 9$$

permutations.

Notice that for each combination, we can count the total number of permutation for that specific combination by finding the number of possible rearrangements. For example $\{a, b\}$ can be counted by

$$\binom{2}{1} = 2$$

as there are 2 ways to rearrange two items (2 items choose the first item, and the second will automatically be chosen). †

As seen above, even for such a small multiset, it is very hard to compute the number of permutations per many combinations for each sized sub-multiset. Thus, exponential generating functions will provide a much easier to compute these permutations.

Example 4.2.7

Given a sequence in which $(a_n = n!)$, or

$$(a_n)_{n=0}^{\infty} = 0!\,, 1!\,, 2!\,, \dots,$$

find its exponential generating function.

Solution. From Definition 4.2.4, we know the EGF of a_n is given by

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

Thus, we substitute $a_n = n!$:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!\,x^n}{n!} &= \sum_{n=0}^{\infty} x^n \\ &= x^0 + x + x^2 + \dots \\ &= 1 + x + x^2 + \dots \\ &= \boxed{\frac{1}{1-x}}. && \text{(Geometric Sum)}\end{aligned}$$

†

Example 4.2.8

Find the exponential generating function of m_n, the number of n-permutations of the multiset $\{4 \cdot a\}$.

Solution. Know that there is only permutations of 0 to 4 possible, as it is impossible to form a permutation of 5 items with 4 objects. Since the objects in the multiset are all indistinguishable, there is only one way to form permutations (where permutations equal the number of combinations). Thus, there is only one way to form a 1-permutation ($\{a\}$), one way to form a 2-permutation ($\{a, a\}$), one way to form a 3-permutation ($\{a, a, a\}$), one way to form a 4-permutation ($\{a, a, a, a\}$), and also one way to form a 0-permutation ($\{\}$) by the Principle of Nothingness.

Thus, the sequence for m_n is

$$\begin{aligned} m_n &= (m_0, m_1, m_2, \dots) \\ &= (1, 1, 1, 1, 1, 0, 0, 0, \dots). \end{aligned}$$

Then, the EGF (let this be denoted by $M(x)$) can be expressed as:

$$\begin{aligned} M(x) &= \sum_{n=0}^{\infty} \frac{m_n x^n}{n!} \\ &= \frac{a_0 x^0}{0!} + \frac{a_1 x^1}{1!} + \frac{a_2 x^2}{2!} + \dots + \frac{a_n x^n}{n!} + \dots \\ &= \frac{1x^0}{0!} + \frac{1x^1}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} + \dots + \frac{0x^n}{n!} + \dots \\ &= \frac{1x^0}{0!} + \frac{1x^1}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \frac{1x^4}{4!}. \end{aligned}$$

†

Example 4.2.9

Find the exponential generating function for h_n, the number of n-permutations of the multiset $\{\infty \cdot a\}$ such that a appears an even number of times.

Solution. As discussed in the previous question, the number of n-permutations of multisets containing the same indistinguishable object can only be permutated once per size of the sub-multiset. Therefore, the sequence of the number of n-permutations of a multiset containing an infinite number of object a can be expressed as

$$(1, 1, 1, 1, \dots).$$

h_n also requires the object a's to appear an even number of times, meaning that only even sub-multisets is counted. Thus,

$$(h_n)_{n=0}^{\infty} = (1, 0, 1, 0, 1, 0, \dots).$$

This alternating sequence could be accomplished by

$$\frac{a_n + b_n}{2},$$

where $a_n = 1, -1, 1, -1, \dots$ and $b_n = 1, 1, 1, 1, \dots$.

Now, we just need to find the EGF form of these:

$$\begin{aligned}\text{EGF}(a_n) &= A(x)\\ &= \sum_{n=0}^{\infty} \frac{m_n x^n}{n!}\\ &= \frac{a_0 x^0}{0!} + \frac{a_1 x^1}{1!} + \frac{a_2 x^2}{2!} + \dots + \frac{a_n x^n}{n!} + \dots\\ &= \frac{1}{0!} - \frac{1x^1}{1!} + \frac{1x^2}{2!} - \frac{1x^3}{3!} + \dots,\end{aligned}$$

$$\begin{aligned}\text{EGF}(a_n) &= B(x)\\ &= \frac{1}{0!} + \frac{1x^1}{1!} + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \dots.\end{aligned}$$

Definition 4.2.2 shows that $b_n = e^x$ and also that $a_n = e^{-x}$ (by plugging in $x = -x$ into the EGF). Thus,

$$e^x + e^{-x} = 2 \cdot \frac{1}{0!} + 2 \cdot \frac{1}{2!}x^2 + 2 \cdot \frac{1}{4!}4^2 + \dots.$$

Then, our desired expression

$$\frac{a_n + b_n}{2} = \frac{e^x + e^{-x}}{2},$$

is equal to

$$\frac{e^x + e^{-x}}{2} = \cdot\frac{1}{0!} + \cdot\frac{1}{2!}x^2 + \cdot\frac{1}{4!}4^2 + \dots.$$

Thus, the generating function of h_n (and let $H(x)$ be its exponential generating function) is given by

$$\begin{aligned}H(x) &= \frac{1}{0!} + \frac{1}{2!}x^2 + \frac{1}{4!}4^2 + \dots\\ &= \frac{e^x + e^{-x}}{2}.\end{aligned}$$

This result using alternating signs of $\frac{e^x+e^{-x}}{2}$ is called "cosh," the hyperbolic version of the cosine function (a circular function). †

But what about distinct subsets?

Example 4.2.10

Find the number of 3-permutations that can be formed from the multiset

$$\{2 \cdot a, 3 \cdot b\}.$$

Solution. Using a non-generating function approach, we combinatorially find that there are three possible combinations:

1. $\{b, b, b\}$,
2. $\{a, b, b\}$,
3. $\{a, a, b\}$.

The number of permutations for the first combination is

$$\begin{aligned}\binom{3}{3} &= \frac{3!}{0!\,3!} \\ &= 1,\end{aligned}$$

as there is no way to rearrange those indistinguishable b's. Next, the number of permutations for the second combination is

$$\begin{aligned}\binom{3}{1} &= \frac{3!}{1!\,2!} \\ &= 3.\end{aligned}$$

The number of permutations for the third is the same by binomial symmetry:

$$\begin{aligned}\binom{3}{2} &= \frac{3!}{2!\,1!} \\ &= 3.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{3!}{0!\,3!} + \frac{3!}{1!\,2!} + \frac{3!}{2!\,1!} &= 1 + 3 + 3 \\ &= 7\end{aligned}$$

3-permutations of the multiset. †

Now a generating functions approach.

Solution. Let a_n represent the sequence of the number of n-permutations of the multiset $\{2 \cdot a, 3 \cdot b\}$ and $A(x)$ be the exponential generating function of a_n, where we want to find the coefficient of $\frac{x^3}{3!}$ in that infinite sum. Note that if we count the number for each number in the sequence in order to find the first few terms of the generating function, it would be the same as the previous solution. Thus, we apply the fact that the property of OGFs, which allows problems to be broken down and the OGF to be obtained by multiplying each of the broken down parts' OGF, also apply to EGFs. This property is called the product of analysis, which will be discussed in more detail and proven after this solution.

We split this into two parts: (1) finding the exponential generating function $B(x)$ for the number of n-permutations of the multiset $b_n = \{2 \cdot a\}$ and (2) finding the exponential generating function $C(x)$ for the number of n-permutations of the multiset $c_n = \{3 \cdot b\}$.

First, b_n, as before, can be represented by the sequence

$$(b_n)_{n=0}^{\infty} = (1, 1, 1, 0, 0, \dots)$$

because there is only one way to obtain each sized multiset (in which the number of combinations per size again equals the number of permutations, equal to 1). Therefore, $B(x)$ can be expressed by

$$B(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2.$$

Using the same logic,

$$c_n = (1, 1, 1, 1, 0, 0, 0, \dots).$$

Thus,

$$C(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

By the product of analysis,

$$\begin{aligned} A(x) &= B(x) \cdot C(x) \\ &= (\frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2)(\frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3). \end{aligned}$$

To obtain the coefficient for $\frac{x^3}{3!}$, we can first look for the coefficient of x^3. By inspection, the only multiplications that can obtain a term with x^3 are:

1. $\frac{1}{0!}$ and $\frac{1}{3!}x^3$,
2. $\frac{1}{1!}x$ and $\frac{1}{2!}x^2$,
3. $\frac{1}{2!}x^2$ and $\frac{1}{1!}x$.

Therefore, the coefficient of x^3 can be spotted in the following representation of the product:

$$B(x) \cdot C(x) = \dots + x^3 \left(\frac{1}{0!} \cdot \frac{1}{3!} + \frac{1}{1!} \cdot \frac{1}{2!} + \frac{1}{2!} \cdot \frac{1}{1!} \right) + \dots.$$

Thus,

$$\begin{aligned} a_3\frac{1}{3!} &= \left(\frac{1}{0!}\cdot\frac{1}{3!}+\frac{1}{1!}\cdot\frac{1}{2!}+\frac{1}{2!}\cdot\frac{1}{1!}\right) \\ &= \frac{1}{0!\,3!}+\frac{1}{1!\,2!}+\frac{1}{2!\,1!}, \end{aligned}$$

or

$$a_3 = \frac{3!}{0!\,3!}+\frac{3!}{1!\,2!}+\frac{3!}{2!\,1!}.$$

This obtains the same form sum as the raw computation in the previous solution, which evaluates the same answer:

$$\begin{aligned} \frac{3!}{0!\,3!}+\frac{3!}{1!\,2!}+\frac{3!}{2!\,1!} &= 1+3+3 \\ &= \boxed{7}. \end{aligned}$$

†

The product we obtained that was the coefficient of $x^3\frac{1}{3!}$ is known as the convolution property of exponential generating functions. This is a result similar to the Cauchy product (the convolution property of OGFs) that is obtained when finding specific coefficients of OGFs.

Theorem 4.2.11 (Cauchy Product)

Named after the French mathematician Augustin-Louis Cauchy, a Cauchy product is the multiplication or discrete convolution of two infinite series. Let the two power series $A(x)$ and $B(x)$ be defined as the following:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, their Cauchy product is given by:

$$C(x) = A(x)\cdot B(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Theorem 4.2.12 (Convolution Property of EGFs)
Given exponential generating functions $A(x)$ and $B(x)$ where

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \quad \text{and} \quad B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!},$$

the convolution property states that

$$C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

These theorems are the core of generating functions that allow us to use simple generating functions to form the whole generating function of more complicated problems formed by the compositions of the simpler sequences. As a result, we are able to find specific a_n's that we do not yet know. These theorems can also be called the Product of Analysis, which leads to a theorem for finding the exponential generating function of multisets:

Theorem 4.2.13 (Product of Analysis)
Let $A_i(x)$ be the exponential generating function for the number of n-permutations of the sub-multiset $\{m_i \cdot d_i\}$ of the multiset

$$S = \{m_1 \cdot d_1, m_2 \cdot d_2, \ldots, m_k \cdot d_k\},$$

where $\sum_{j=0}^{n} m_j = |S|$. Then, the exponential generating function for the number of n-permutations of multiset S is given by the product of the exponential generating function of each (maximum-sized) indistinguishable sub-multiset:

$$\begin{aligned} A(x) &= \prod_{i=0}^{n} A_i(x) \\ &= A_1(x) \times A_2(x) \times \cdots \times A_n(x). \end{aligned}$$

Before we begin the proof of for Theorem 4.2.13, another theorem needs to be mentioned that counts the number of permutations of a multiset.

Theorem 4.2.14 (Number of Permutations of a Multiset)
The number of n-permutations of an n-sized multiset is given by

$$\frac{n!}{p_1!\,p_2!\cdots p_n!},$$

where p_i is the multiplicity of an object i within the multiset.

Proof. The number of n-permutations of an n-multiset is equal to the number of rearrangements possible with those n items. If all items are distinct, then the number of rearrangements is given by n choices for the first spot, $n-1$ for the second, and all the way down to 1 choice for the nth slot of the multiset:

$$n \times n-1 \times n-2 \cdots 2 \times 1 = n!\,.$$

This is also true due to the permutations formula:

$$\begin{aligned} P_n^n &= \frac{n!}{(n-n)!} \\ &= \frac{n!}{0!} \\ &= n!\,. \end{aligned}$$

Let p_i represent the multiplicity of an object i, in which $p_1+p_2+\cdots+p_n = n$. Then, for any indistinguishable object, overcounting occurs, and $n!$ is divided by the possible rearrangements upon those p_i indistinguishable objects i to acount for overcounting. Thus, the number of permutations of a multiset of any n objects is

$$\frac{n!}{p_1!\,p_2!\cdots p_n!}.$$

□

Now, we start the proof for Theorem 4.2.13:

Proof. Let the exponential generating function for the number of permutations of a multiset $S = \{m_1 \cdot d_1, m_2 \cdot d_2, \ldots, m_k \cdot d_k\}$ be given by $A(x)$, where

$$A(x) = \frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \ldots.$$

Additionally, suppose that $H_i(x)$ is the generating function for counting permutations (h_i for the number for a multiset of size i) of the multiset

$$\{m_i \cdot d_i\},$$

where object d_i has multiplicity m_i in set S.

Since every set that $H_i(x)$ is a generating function of only has m_i indistinguishable objects, h_i is given by an indicator variable (a result of 0 or 1) on whether or not a j-sized sub-multiset can consist of j objects d_i, where $0 \leq j \leq m_i$. Thus,

$$h_i = 1, 1, 1, \ldots, m_i, 0, 0, 0, \ldots$$

and

$$H_i(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{m_i!}x^{m_i}.$$

Note that this can be alternatively be represented by the indicator variable δ_r in which $\delta_r = 1$ if a permutation of r objects is possible and 0 otherwise:

$$H_i(x) = \frac{\delta_0}{0!} + \frac{\delta_1}{1!}x + \frac{\delta_2}{2!}x^2 + \cdots + \frac{\delta_{m_i}}{m_i!}x^{m_i} + \ldots.$$

Either way, in order to obtain an x^n term in the product, it is required to select and multiply non-zero terms with powers that add up to n:

$$p_1 + p_2 + \cdots + p_k = n,$$

where k represents the number of distinct objects, a permutation with p_f object f's are possible, and $0 \leq p_f \leq m_i$. Then, the product of all the terms with these p_f powers is equivalent to the following:

$$\begin{aligned}\prod_{g=1}^{k} \frac{1}{p_g!}x^{p_g} &= \left(\frac{1}{p_1!}x^{p_1}\right)\left(\frac{1}{p_2!}x^{p_2}\right)\cdots\left(\frac{1}{p_k!}x^{p_k}\right) \\ &= \frac{1}{p_1!\,p_2!\cdots p_k!}x^n.\end{aligned}$$

Therefore, the coefficient of x^n will be the summation of

$$\frac{1}{p_1!\,p_2!\cdots p_k!},$$

over all possible p_f's (such that $p_1 + p_2 + \cdots + p_k = n$ and that a n-permutation with p_1 object 1, p_2 of object 2, ..., and p_k object k possible):

$$\sum_{\substack{p_1+p_2+\cdots+p_k=n \\ p_1,p_2,\ldots,p_k \geq 0}} \frac{1}{p_1!\,p_2!\cdots p_k!}.$$

Using these sums as coefficients of $A(x)$, we obtain an OGF-looking definition:

$$A(x) = v_0 + v_1x + v_2x^2 + \ldots,$$

where each coefficient

$$v_l = \sum_{\substack{p_1+p_2+\cdots+p_k=n \\ p_1,p_2,\ldots,p_k \geq 0}} \frac{1}{p_1!\,p_2!\cdots p_k!}.$$

Comparing this to the original definition of $A(x)$ in the beginning that defined

$$A(x) = \frac{a_0}{0!} + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \dots,$$

we get an expression for a particular term in the sequence $(a_n)_{n=0}^{\infty}$:

$$\frac{a_n}{n!} = \sum_{\substack{p_1+p_2+\dots+p_k=n \\ p_1,p_2,\dots,p_k \geq 0}} \frac{1}{p_1!\,p_2!\cdots p_k!}.$$

Multiplying both sides by $n!$ then obtains an expression for the number of permutations of a multiset S in which $|S|= n$, a_n:

$$a_n = \sum_{\substack{p_1+p_2+\dots+p_k=n \\ p_1,p_2,\dots,p_k \geq 0}} \frac{n!}{p_1!\,p_2!\cdots p_k!}.$$

Recall Theorem 4.2.14—this result is equivalent to the proven result of that theorem. Therefore, the method of multiplying generating functions $H_i(x)$ for each distinct object and their multiplicities m_i obtains the exact same generating function $A(x)$ for the entire multiset S (that is, each summed coefficient of the infinite series for $A(x)$ was generated through the product of all h_i's):

$$\begin{aligned} A(x) &= \prod_{i=0}^{n} h_i(x) \\ &= h_1(x) \times h_2(x) \times \cdots \times h_n(x). \end{aligned}$$

And thus, the convolution property of exponential generating functions (Theorem 4.2.12) is also proven as a consequent to this proof. □

Exercises 4.2

Exercise 4.2.1

Count the number of permutations of the multiset

$$\{A, L, E, X, S, W, O, R, L, D, O, F, M, A, T, H\}.$$

Exercise 4.2.2

Derive an exponential generating function for the number of n-permutations that exists from a multiset

$$\{2 \cdot A, 3 \cdot B, 4 \cdot C\}.$$

Exercise 4.2.3

Find the number of 4-permutations from the multiset

$$\{3 \cdot D, 2 \cdot E, 4 \cdot F\}.$$

Exercise 4.2.4

Obtain an exponential generating function for the number of n-permutations of $\{\infty \cdot G, \infty \cdot H, \infty \cdot I\}$ such that there are an odd number of G's and an even number of H's, and an even number of I's.

Exercise 4.2.5

Find an exponential generating function for the number of n-permutations of $\{\infty \cdot G, \infty \cdot H, \infty \cdot I\}$ such that there are an odd number of G's (and is at least 2) and an even number of H's (and is at most 6), and an even number of I's (and is at least 3).

Exercise 4.2.6

Find the number of ways to paint 10 distinct boxes such that there can be at most 3 boxes painted red, at most 2 painted orange, at most 1 painted yellow, and any number of boxes painted green or blue.

Exercise 4.2.7

Recall the proof to Theorem 4.2.13. Using a similar proof, structure a proof to the Cauchy product (Theorem 4.2.11).

4.3 Catalan Numbers

Definition 4.3.1 (Catalan Numbers)

Let C_n denote the n-th Catalan number and $C_0 = 1$. Then, for $n \geq 1$, the n-th Catalan number is defined by the following recurrence relation:

$$\begin{aligned} C_n &= \sum_{k=0}^{n-1} C_k C_{(n-1)-k} \\ &= C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0. \end{aligned}$$

Example 4.3.2

List the first five Catalan numbers.

Solution.

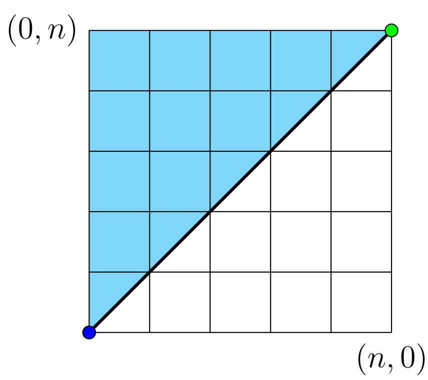
$$\begin{aligned}
C_0 &= 1,\\
C_1 &= C_0C_0\\
&= 1,\\
C_2 &= C_0C_1 + C_1C_0\\
&= 1\cdot 1 + 1\cdot 1\\
&= 2,\\
C_3 &= C_0C_2 + C_1C_1 + C_2C_0\\
&= 1\cdot 2 + 1\cdot 1 + 2\cdot 1\\
&= 2 + 1 + 2\\
&= 5,\\
C_4 &= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0\\
&= 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1\\
&= 5 + 2 + 2 + 5\\
&= 14.
\end{aligned}$$

Thus, the first five Catalan numbers are $1, 1, 2, 5, 14$. †

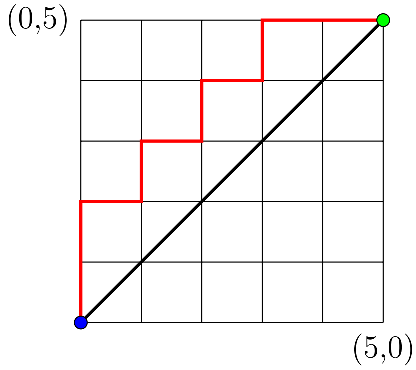
Definition 4.3.3

Dyck Paths A Dyck path of height n and a total length of $2n$ of a $n \times n$ grid is a lattice path from $(0, 0)$ to (n, n) such that the path always stays on or above the $y = x$ diagonal and only right and up moves are allowed. Alternatively, this is the number of lattice paths of a right triangle from one of the non-right angled points to the other.

The image below shows the region in which the starting dot at $(0, 0)$ may enter:



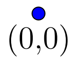
Here is an example Dyck Path of a 5×5 grid:



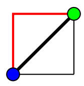
Example 4.3.4

Count the number of Dyck paths of height 0 to 3.

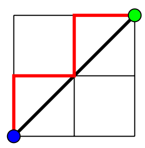
Solution. Let D_n be the number of Dyck paths of height n. Then, $D_0 = 1$ (Principle of Nothingness):

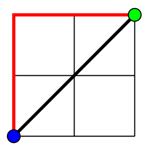


Next, for a path of height 1, only one such Dyck paths can be constructed ($D_1 = 1$):

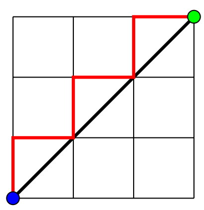


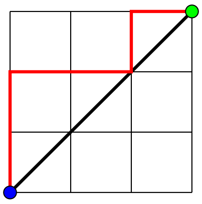
For a path of height 2, two such Dyck paths can be constructed ($D_2 = 2$):

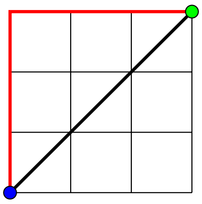


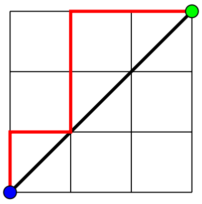


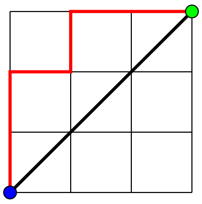
Lastly, a Dyck path of height 3 would be possible in 5 different ways ($D_3 = 5$):











†

Notice that for these first few terms ($0 \le n \le 3$) for the number of Dyck paths matches the first few Catalan numbers. Then, the following proposition is formed.

Proposition 4.3.5

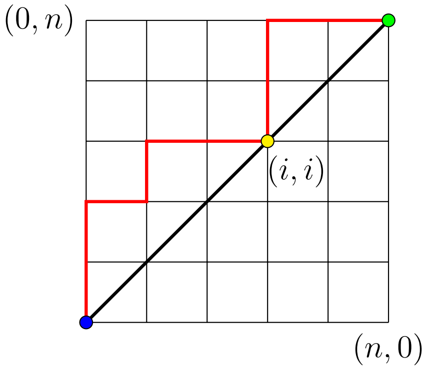
The number of Dyck paths of height n, D_n, is given by the Catalan number C_n. That is, for any n,

$$D_n = C_n.$$

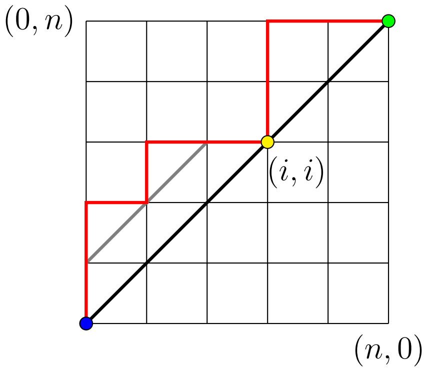
This also follows

$$\begin{aligned} D_n &= \sum_{k=0}^{n-1} D_k D_{(n-1)-k} \\ &= D_0 D_{n-1} + D_1 D_{n-2} + D_2 D_{n-3} + \cdots + D_{n-1} D_0. \end{aligned}$$

Proof. Consider an arbitrary Dyck path of height n that intersects the diagonal again for the first time at point (i, i):



Then, the path to (i, i) can be counted by the Dyck path D_{i-1} because this is the first intersection of the diagonal, meaning there cannot be any other intersection before it. Thus, a Dyck path from $(0, 1)$ to $(i-1, i)$ guarantees such case in which the next path is already chosen to be right (and the first move is up, as per all Dyck paths or else it would go over the diagonal):



The number of possible remaining paths then can be directly counted by the number of Dyck paths from (i, i) to (n, n) since there are no other restrictions to the number of times the path will touch the diagonal. This is simply given by the number of Dyck paths of length $n - i$: D_{n-i}.

Note that this method also accounts for the cases in which the path "never" touches the diagonal other than at the last point (n, n): (n, n) would be the first point in which the point returns to the diagonal, meaning that it counts D_{n-1} such paths and since there are no more moves, the remaining path can be counted by D_0, which satisfies our statement.

Thus, since these two paths are independent from one another, the multiplication principle states that there are

$$D_{i-1}D_{n-i}$$

such paths. Summing over all the possible values of i ($1 \leq i \leq n$—since $i-1$ and $n-i$ cannot be negative) gives us the following formula:

$$D_n = \sum_{i=1}^{n} D_{i-1}D_{n-i}.$$

Performing an index shift of $k = i - 1, i = k + 1$:

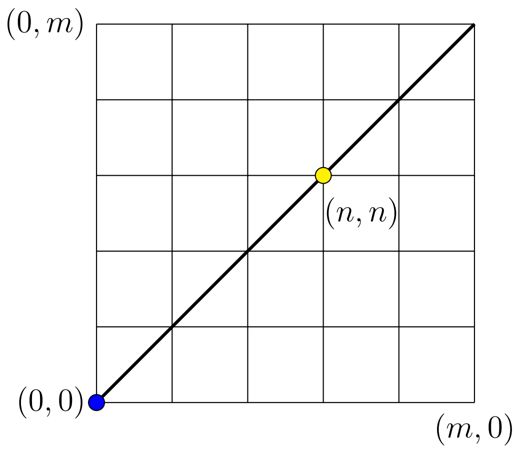
$$
\begin{aligned}
D_n &= \sum_{i=1}^{n} D_{i-1}D_{n-i} \\
&= \sum_{k+1=1}^{n-1} D_{(k+1)-1}D_{n-(k+1)} \quad \text{(Index Shift, including } n \text{ to maintain same index range)} \\
&= \sum_{k=0}^{n-1} D_k D_{n-k-1} \\
&= D_0D_{n-1} + D_1D_{n-2} + D_2D_{n-3} + \cdots + D_{n-1}D_0.
\end{aligned}
$$

Therefore, a bijection has been proven between the number of Dyck paths of height n with the n-th Catalan number. □

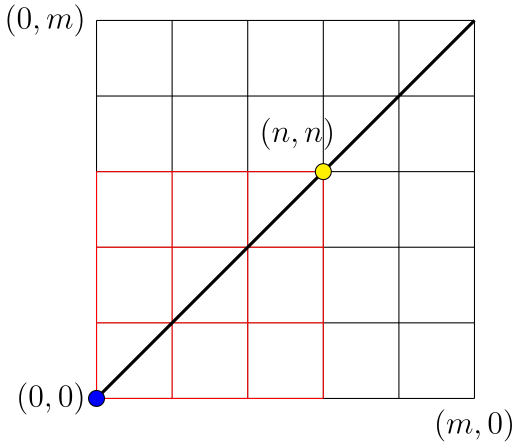
Theorem 4.3.6 (Closed Formula for Catalan Numbers)
Other than a recursion, the n-th Catalan number can also be obtained through a closed formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proof. Consider the following graph, in which $m > n, m, n \notin \mathbb{Z}^-$:



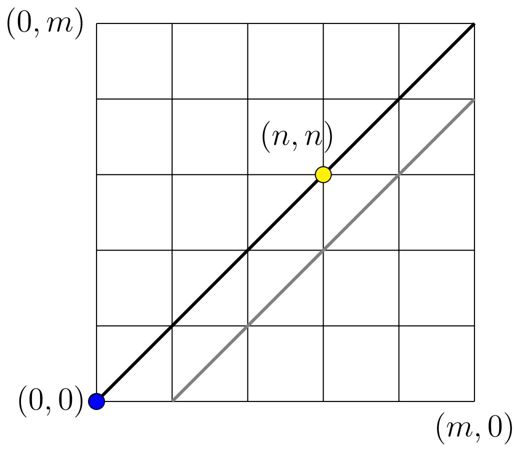
The number of paths from $(0,0)$ to (n,n) (in the region as shown in red) contains some number of paths in which the path only stays above the diagonal $y = x$, counting the number of Dyck paths:



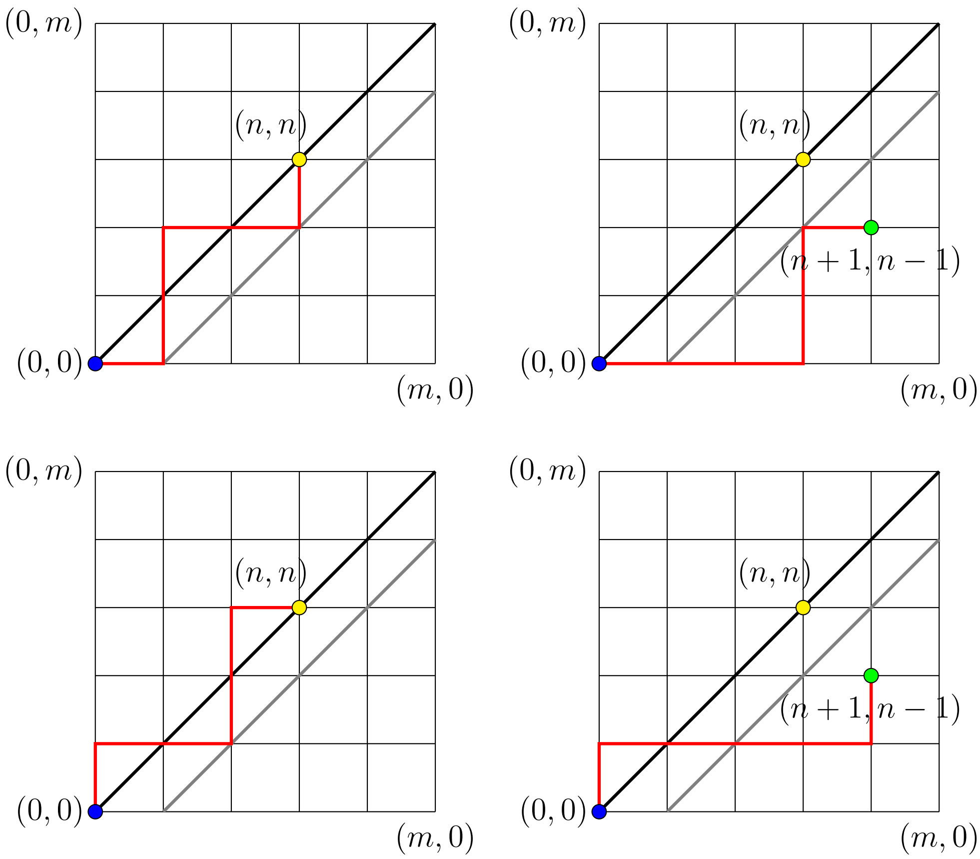
The number of total lattice paths from $(0, 0)$ to (n, n) is equal to the total moves $(2n)$ choosing the number of up moves or the number of right moves, which will automatically choose the rest of the moves:

$$\binom{2n}{n}.$$

Next, the number of paths that are not Dyck paths but are counted in that total number of lattice paths all touch the line $y = x - 1$, as shown below in gray:



The line $y = x - 1$ always intersects with the lattice paths that do not stay on or above the line $y = x$ because they take one or more right after touching $y = x$. Thus, let the rest of the path after the first intersection of that invalid path with $y = -x$ be reflected across $y = -x$. Two of such reflections are shown below:



Notice that both of these reflections end up at $(n+1, n-1)$? It is true that for any reflection of a possible lattice path from $(0,0)$ to (n,n) across $y = x - 1$, the last point will be at $(n+1, n-1)$. This is because (n,n) reflected across $y = x - 1$ yields the y point of the old $x - 1$, which is $n - 1$. Similarly, the new x point will be equal to the old y added to 1, since $x = y + 1$, which obtains $n + 1$.

Any lattice path formed from $(0,0)$ to $(n+1, n-1)$ must cross the line $y = x - 1$, and also note that this reflection is invertible (by the definition of reflections). Since the reflection from a path from $(0,0)$ to $(n+1, n-1)$ is invertible (and vice versa), there must be no duplicates for either these paths for the invalid Dyck paths. Therefore, each of these paths to $(n+1, n-1)$ must map onto a unique invalid Dyck path, and the number of reflected lattice paths from $(0,0)$ to $(n+1, n-1)$ has a one-to-one correspondence, or a bijection, with the number of invalid Dyck paths from $(0,0)$ to (n,n). The number of invalid Dyck paths, then, is given by the total number of possible lattice paths from $(0,0)$ to $(n+1, n-1)$ is given by the total number of lattice paths choosing either the number of right moves $(n+1)$ or choosing the number of up moves $(n-1)$ in which the rest will be automatically chosen:

$$\begin{aligned}\binom{n+1+n-1}{n+1} &= \binom{2n}{n+1}\\ &= \binom{2n}{n-1}. && \text{(Binomial Symmetry)}\end{aligned}$$

Therefore, since the total number of lattice paths from $(0,0)$ to (n,n) is

$$\binom{2n}{n}$$

and the number of invalid Dyck paths is given by

$$\binom{2n}{n+1} = \binom{2n}{n-1},$$

the number of valid Dyck paths from $(0,0)$ to (n,n) is equal to the following by complementary counting:

$$\binom{2n}{n} - \binom{2n}{n-1}.$$

Converting these binomial coefficients into factorials using the combinations formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!},$$

we obtain Theorem 4.3.6:

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!\,(2n-n)!} - \frac{(2n)!}{(n-1)!\,(2n-(n-1))!} \\
&= \frac{(2n)!}{n!\,(n)!} - \frac{(2n)!}{(n-1)!\,(n+1)!} \\
&= \frac{(2n)!}{n!\,(n)!} - \frac{(2n)!}{(n-1)!\,(n)!\,(n+1)} \\
&= \frac{(2n)!}{n!\,(n)!} - \frac{(2n)!\,(n)}{(n)!\,(n)!\,(n+1)} \\
&\qquad \text{(Multiply by } \tfrac{n}{n} \text{ to combine with } (n-1)!) \\
&= \left(1 - \frac{n}{(n+1)}\right) \frac{(2n)!}{n!\,(n)!} \\
&= \left(1 - \frac{n}{n+1}\right) \binom{2n}{n} \\
&= \left(\frac{n+1}{n+1} - \frac{n}{n+1}\right) \binom{2n}{n} \\
&= \frac{n+1-n}{n+1} \binom{2n}{n} \\
&= \boxed{\frac{1}{n+1} \binom{2n}{n}}.
\end{aligned}$$

□

Exercises 4.3

Exercise 4.3.1

Find the number of lattice paths from $(0,0)$ to $(5,5)$ are possible such that the path always stays on or below the diagonal.

Exercise 4.3.2

Suppose there are two candidates in an upcoming election for the AWM city counsel: David and Candice. Let there be $2n$ voters, where n voters are voting for David while n other voters are voting for Candice. All voters form a line to vote and the ballots are counted one at a time. Find the probability that for any point in time, Candice's votes are never greater than those of David.

Exercise 4.3.3

Use a generating functions approach (OGFs suggested) to prove and result in the closed formula from Theorem 4.3.6 for the n-th Catalan number.

Exercise 4.3.4

Let $w(n)$ be a function of a closed-expression of n such that

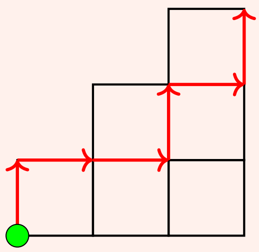
$$C_{n+1} = w(n)C_n,$$

where C_n denotes the n-th Catalan number. Use this expression of $w(n)$ and $C_0 = 1$ to compute the values for

$$C_1, C_2, \ldots, C_6.$$

Exercise 4.3.5

Consider a staircase formed by square blocks as shown below (of size 3×3). Suppose you may only move up or right along the staircase-like structure, and that all paths are of length 6 (such as the path with six steps indicated by the arrows below), find the number of paths in a $n \times n$ staircase.



Exercise 4.3.6

$2n$ people meet around a circular table to form a consensus. Assume that every person at that table have very long arms so that they can shake hands with anyone else at the table. Everyone decides to shake hands simultaneously. In how many ways is it possible for this to happen such that there are no intersecting handshakes (i.e. crossing handshakes)?

Exercise 4.3.7

Suppose you have $n-1$ diagonals to divide triangulations of a convex $(n+2)$-sided polygon, or a $n+2$-gon, into triangles with disjoint interiors. Prove that the number of such cases is given by the n-th Catalan number. *Hint: What is the total number of segments, which include both sides and diagonals?*

Exercise 4.3.8

Let $f(n)$ be the number of sequences of integers $(a_k)_{k=1}^n$ such that $a_k \geq -1$. Additionally, suppose that all partial sums of this sequence are non-negative:

$$S_k = a_1 + a_2 + \cdots + a_k, \forall S_k \geq 0, S_k \in \{1, 2, \ldots, n\}.$$

If

$$a_1 + a_2 + \cdots + a_n = 0,$$

prove that $f(n) = C_n$, where C_n denotes the n-th Catalan number.

CHAPTER 5

Recurrence Relations

"Mathematics is for lazy people" - Peter Hilton (British Mathematician)

To add up a finite geometric sum, we could either do it by hand or by using the handy formula. We choose to do it the lazy way, by creating and using awesome formula. But what do these formulas even mean, and how do you derive them? Today, I will introduce you to recurrence relations, a topic useful across many applications in combinatorics, discrete mathematics, and data structures. Recurrence relations deal with sequences of data and can significantly help in simplifying complex problems.

Definition 5.0.1 (Recurrence Relations)

A recurrence relation is an equation that recursively defines a sequence, where each term of the sequence can be obtained by utilizing previous terms. Generally, a recurrence relation for a sequence $\{a_n\}$ takes the following form:

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}),$$

where f is some function and k is the number of terms that are needed to define a_n. The entire sequence can typically be determined when the first k values $a_0, a_1, \ldots, a_{k-1}$ are specified.

An example of a recurrence relation with constant coefficients is shown below:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are constants. This structure should be familiar/commonly appear in the next few sections.

In this chapter, we will explore multiple methods to solve linear recurrences, whether it be homogeneous or non-homogeneous:

1. Characteristic Root Method, Homogeneous
2. Characteristic Root Method, Non-Homogeneous
3. Generating Functions Method
4. Proof by Induction Method
5. Iteration/Back-Substitution Method
6. Recursion Tree Method

We will first solve some example questions using the Characteristic Root Method, then we will briefly look at the other methods. After this, we will jump into some extra, interesting problems. There are exercise problems are at the end of each of these methods' sections as usual. Linear recurrences will be defined in just a bit, but please note that non-linear recurrences will not be focused in this chapter due to their complexity.

5.1 Characteristic Root Method: Homogeneous

$$a_n = \begin{cases} a_1 & \text{if } n = 1 \\ a_{n-1} + d & \text{if } n > 1 \end{cases}$$

$$a_n = \begin{cases} a_1 & \text{if } n = 1 \\ a_{n-1} \cdot r & \text{if } n > 1 \end{cases}$$

Remember these recursive formulas for arithmetic and geometric series you learned in your high school career? Great! Those are common examples of recurrence relations. Initially, recurrence relations may seem unnecessary—at least, that's what I thought when I first learned about them. I wondered why we needed them if we had explicit formulas that could directly spit out any term. However, I soon realized I was wrong.

Recurrence relations allow us to describe problems with a new representation (like the famous Tower of Hanoi Puzzle, which will be shown later) and break problems into simpler steps.

Even cooler, you could use this to obtain explicit formulas like for the Fibonacci Sequence through many different techniques (you may want to check out the generating functions topic in the beginning if you haven't, where we used generating functions to do so; In this section, we will apply other techniques). But do not fret—We will solve many recurrence relations in this chapter. First, we'll go through some definitions, and then we'll dive into some example problems to solve these recurrences using the Characteristic Root Method.

The recursive formulas of the arithmetic and geometric series were specifically linear, first-order recurrence relations. What do they mean?

Definition 5.1.1 (Linear Recurrences)
A recurrence relation is linear when the recurrence terms of function a are linear. So

$$a_n = 3a_{n-1} + a_{n-2}$$

and arithmetic sequences $a_n = a_{n-1} + d$, where $n > 1$, are both linear recurrence relations, as the a_{n-k} terms are linear (whereas $a_n = a_{n-1}(a_{n-1} + 2)$ is not, and is specifically a quadratic recurrence relation).

Definition 5.1.2 (k-th Order Recurrences)
kth-order recurrence relations are recurrence relations with its "deepest" recurrence term being a_{n-k}. In other words, that is the number of terms you would have to evaluate to determine the next term in the sequence, in this case, k terms. For example,

$$a_n = a_{n-3} + a_{n-4}$$

is a fourth-order recurrence relation since you need to go back and find four previous values from a_{n-1} until a_{n-4} to obtain the next value of a_n.

Let's look at the arithmetic sequence for $n > 1$:

$$a_n = a_{n-1} + d.$$

This equation we already know is a first-order, linear recurrence relation. But there is another classification that you should know about:

Definition 5.1.3 (Homogeneous vs. Non-Homogeneous)
A recurrence relation is homogeneous when there are no constant terms besides the recurrence terms. The arithmetic sequence formula for $n > 1$ has a constant term d, therefore it is classified as non-homogeneous.

On the other hand, the geometric sequence recursive formula where $n > 1$,

$$a_n = a_{n-1} \cdot r,$$

is homogeneous since it has no constant terms, as r is a constant coefficient of a recurrence term.

Note that $2n$ or any term of n that is not "$an - k$" terms counts as a constant, meaning that

$$a_n = a_(n-1) + 10n$$

would be an example of a non-homogeneous, linear recurrence relation.

Let's start by demonstrating how to solve simpler recurrence relations: linear homogeneous ones. Note that in this chapter, "solving" these recurrence relations equates to finding explicit formulas.

Example 5.1.4

Let the sequence of a_n be defined as

$$a_n = a_{n-1} + 2a_{n-2},$$

where $a_0 = 1$ and $a_1 = 5$. Find an explicit formula for this recursive sequence.

The way we will use to solve this would be to use the characteristic polynomial approach (or "Characteristic Root Method").

Definition 5.1.5 (Characteristic Root Method)

The characteristic root method is a technique designed to solve linear homogeneous recurrence relations by finding roots of the recurrence's characteristic equation. These roots provide a general solution in exponential terms for the recurrence, which can then be specified to a particular solution through given initial values to solve for constants.

This method follows these rules for this homogeneous linear recurrence:

1. Re-arrange the equation and use the lambda hypothesis to find the characteristic polynomial
2. Evaluate the factors of the characteristic polynomial
3. Set up a general solution from the factors
4. Solve for coefficients to find the specific solution by plugging in given information (note: you need at least k given points (n, a_n) to do so for recurrences with k a_{n-x} terms)

Definition 5.1.6 (Characteristic Polynomial)

A characteristic polynomial of a linear homogeneous recurrence relation is the part of the function's factored form that gives non-zero roots when set to 0. The other part of the factored form would be in the form of a general solution.

On a general level, the characteristic polynomial of the kth-order sequence

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$$

is

$$\lambda^k - c_1\lambda^{k-1} - \cdots - c_k = 0,$$

for c are constant coefficients.

This may be a little confusing at first, but it will make more sense once we dive in to the problems.

Solution. To start, we need to rearrange the equation all to one side set equal to 0. Doing this, we obtain

$$a_n - a_{n-1} - 2a_{n-2} = 0.$$

From this, we apply the lambda hypothesis (not commonly used terminology, but will be used in this chapter; r can also be used in place of lambda), which denotes setting

$$a_n = \lambda^n$$

as a hypothesis of its solution. This works for linear homogeneous recurrences. Substituting, we get:

$$\lambda^n - \lambda^{n-1} - 2\lambda^{n-2} = 0$$

and through the distributive property, we can rewrite this as

$$\lambda^{n-2}(\lambda^2 - \lambda - 2) = 0.$$

The first part, λ^{n-2}, is in the form of the general solution while the polynomial of lambda in the second part is the characteristic polynomial, which we can proceed to factor and obtain roots from, thus its name of a "characteristic" polynomial. The general solution does not provide any useful information.

So we have

$$\lambda^{n-2}(\lambda - 2)(\lambda + 1) = 0,$$

meaning $\lambda = 2$ and $\lambda = -1$ as the two roots of this second-order linear recurrence. This also means that

$$\lambda^n = 2^n$$

and

$$\lambda^n = (-1)^n$$

are functions for the two roots.

The linear recurrence function can then be expressed in the form

$$a_n = A(2^n) + B(-1)^n.$$

Note that this is somewhat similar to breaking down partial fractions after finding the factored form.

Now, we can simply use the two initial conditions to solve for A, B and we are done! This is also why we need at least k given initial points or conditions in order to solve a system of equations at the end of these problems. We have $a_0 = 1, a_1 = 5$, so we get the two equations:

$$1 = A + B,$$

$$5 = 2A - B.$$

Solving by elimination:

$$6 = 3A, A = 2,$$

$$1 = 2 + B, B = -1.$$

$\therefore$ we finally obtain the explicit formula solution for this recurrence:

$$a_n = 2 \cdot 2^n + -1(-1)^n = \boxed{2^{n+1} + (-1)^{n+1}, \forall n \geq 0}.$$

Let's test a value to see whether this works or not:

$$a_n = a_{n-1} + 2a_{n-2} = 2^{n+1} + (-1)^{n+1}, \text{ where } a_0 = 1 \text{ and } a_1 = 5$$

So

$$\begin{aligned} a_2 &= a_1 + 2a_0 \\ &= (5) + (2)(1) \\ &= 7 \end{aligned}$$

by the given recurrence formula. Using our formula, we get

$$\begin{aligned} a_2 &= 2^3 + (-1)^3 \\ &= 8 - 1 = \boxed{7} \checkmark \end{aligned}$$

†

Wonderful—what a spectacular and fun way to obtain an explicit formula given a recursive formula!

In the next problem, we will see what happens when a linear homogeneous recurrence a_n's characteristic polynomial has two or more same roots; That is, having a multiplicity m_i of $m_i \geq 2$.

Example 5.1.7

The following equation gives a recurrence for a_n:

$$a_n = 6a_{n-1} - 9a_{n-2}.$$

Given that $a_0 = 2$ and $a_1 = 9$, Find an explicit formula for a_n.

Applying the same steps as the previous question, we get

$$a_n - 6a_{n-1} + 9a_{n-2} = 0,$$

$$\lambda^n - 6\lambda^{n-1} + 9\lambda^{n-2} = 0,$$

$$\begin{aligned} \lambda^{n-2}(\lambda^2 - 6\lambda + 9) &= \lambda^{n-2}(\lambda - 3)^2 \\ &= 0, \end{aligned}$$

$$\lambda = 3, 3.$$

So we have the characteristic roots 3,3 and so $\exists$ ("there exists") 3^n functions in the general solution form:

$$a_n = A(3^n) + B(3^n)$$

Condition 1:

$$2 = A + B$$

Condition 2:

$$9 = 3A + 3B, 3 = A + B?$$

This does not make sense, it is impossible to obtain values for A, B as these two equations are parallel lines. Additionally, the general form can be rewritten as

$$a_n = A(3^n) + B(3^n) = 3^n(A + B),$$

which expresses a geometric sequence. But our sequence is definitely not geometric since geometric recurrence formulas are just

$$a_n = r{\cdot}a_{n-1}.$$

So what went wrong?

First off, notice that each root in the general form turns into its own function in the form r^n, where r is each characteristic root. The form that we used for the general solution was incorrect, causing us to end up with a system of equations that had no solutions.

In the previous problem, this form worked because the roots were distinct. But in this problem, we have two equal roots, meaning one root with a multiplicity of 2, and there was a problem.

Therefore, when dealing with characteristic roots r with multiplicity $m \geq 2$, we must follow a rule called linear independence to ensure a correct solution space.

Definition 5.1.8 (Linear Independence)

Two functions f and g are linearly independent if they can be expressed in the following form:

$$f = rg,$$

where $r \in \mathbb{R}$ is a constant.

This term is actually a concept under linear algebra and differential equations, where linear independence is also defined over a set of vectors/vector matrices.

But how do we make sure that two functions are linearly independent? Let me provide you with some examples.

Take the function $f = 3x$. This function has x_1 in its term along with a constant. That means any other function g with the variable x and a constant coefficient like $5x$

would be linearly dependent on function f. To make these two functions have linear independence, we can change the power of x, namely making either f or g have an extra x. Having an extra x would ensure that these functions cannot be mapped onto to another through a series of real number combinations of the others. For example, if you had $3x$ and $5x^2$, there is no way to finitely map $3x$ to $5x^2$ by adding or subtracting combinations of $3x$.

Therefore, $3x$ and $5x^2$ are linearly independent. As mentioned before, distinct exponential functions would already be linearly independent, but the same exponential functions would be linearly dependent.

But why do we even need this in the first place? Our solution set should be 2-dimensional in our case because we have 2 roots due to the sequence being second-order. If we have linearly dependent functions in the general form, we would be able to express

$$a_n = B(f) + A(g)$$

as

$$a_n = y(C \cdot A + D \cdot B),$$

where y is a function and $A, B, C, \text{and} D$ are constants.

In other words, we would be able to express an as $y(E)$ for E is a constant since $A, B, C, \text{and} D$ were all constants as well. This would mean that our second-order sequence is a first-order one with 1-dimensional solution space as there is only one term $y(E)$, which is invalid.

Therefore, to have the same amount of solution space dimensions as the order, we need to make sure each function produced by the factor of the characteristic roots cannot be expressed as another function by multiplying by constants.

To do this, we can make sure every root has a bijection (recall this to be a one-to-one and onto mapping) to a function that is linearly independent of any other within the same multiplicity group (a group of m roots with the same multiplicity m), where we multiply by n each time to shift the solution space to our desired m-dimensions per multiplicity group:

For a kth-order recurrence

$$\exists x \in \{1, 2, 3, \ldots, k\},$$

and each root r_x of multiplicity m_x is assigned a unique function within the set of linearly independent functions,

$$S = r_x^n, nr_x^n, n^2r_x^n, n^3r_x^n, \ldots, n^{m-1}r_x^n.$$

This bijection ensures that a root r_x with multiplicity m is assigned to a correct solution space. It is correct that for a second-order recurrence with a characteristic root with a multiplicity of 2, $n^2r_x^n$ and r_x^n would be linearly independent. However, we must not overspecify the solution space, as doing so would hint at a multiplicity of 3 and be incorrect. So we just have to follow a correspondence with S, which only has the bijection of powers of n to the power of $m-1$, ensuring that we do not go beyond the range of this sequence's solution space.

So a distinct root would just have r_x^n correspond to it. For a root r_x^n with multiplicity 2, the two linearly independent functions that would correspond to the roots would be $r_x^n, n(r_x^n)$. A multiplicity 3 would have $r_x^n, n(r_x^n), n^2(r_x^n)$, and so on.

So if you have all distinct characteristic roots

$$1, 2, 3, 4, 5, 6, \ldots$$

then you have the general solution form as

$$a_n = A_1(1^n) + A_2(2^n) + A_3(3^n) + \cdots + A_k(k^n).$$

But if you had all the same roots of say 2 then you would have

$$a_n = A_1(2^n) + A_2 n(2^n) + A_3 n^2(2^n) + \cdots + A_k n^{k-1}(2^n),$$

which in this case $k = m$. If we had 2, 2, 3 as the characteristic roots, we would have root $r_1 = 2$ with $m = 2$ corresponding to $\{r_1^n, nr_1^n\}$ and root $r_2 = 3$ with $m = 1$ corresponding to $\{r_1^n\}$: a general solution for this would be in the form of

$$a_n = A(r_1^n) + B_n(r_1^n) + C(r_2^n) = A(2^n) + Bn(2^n) + C(3^n).$$

Putting it all together, we work out this surprisingly big formula that accounts for linear independence, an explicit formula from <u>any</u> kth root linear homogeneous recurrence:

Theorem 5.1.9 (Formula for kth Root Homogeneous Linear Recurrence)
The explicit formula for any kth root homogeneous linear recurrence can be found through the following formula:

$$a_n = \sum_{x=1}^{z} \sum_{k=0}^{m_x - 1} A_{(x,k)} n^k r_x^n,$$

where there are z distinct roots r_x of multiplicity m_x, and a constant-coefficient $A(x, k)$ for all terms.

This big expression may seem daunting, but I assure you it is not. The outer sum accounts for all "z" multiplicity groups and the inner sum denotes each root within each multiplicity group, with each term's constant-coefficient Ak, $n^{m_x - 1}$ which ensures linear independence, and the r_x^n function from our characteristic roots.

Also, notice that we have already used this equation in the previous problem where we had distinct roots (meaning multiplicities $m = 1$), and we had n_0 hidden in each term and the rest works out. Breaking the equation down into pieces, you will realize that it all works out pretty well.

Solution. Going back to the original question,

$$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 2 \text{ and } a_1 = 9,$$

we have already found the characteristic roots to be 3, 3. Now we know that a root $r_1 = 3$ of multiplicity $m = 2$ would have the general solution form

$$\begin{aligned} a_n &= Ar_1n + Bnr_1n \\ &= A(3n) + Bn(3n). \end{aligned}$$

The formula would also give us the same equation with 1 distinct root r_1 of multiplicity 2 and constant coefficients A_1 and A_2:

$$a_n = A_1(3n) + A_2n(3n),$$

basically the same equations using different letters to denote the same constant.

Now, plug in using our given two initial conditions to obtain a system of equations to solve for A_1 and A_2:

$$\begin{gathered} 2 = A_1 \\ 9 = A_1(3) + A_23, 3 = A_1 + A_2 \end{gathered}$$

Substituting,

$$3 = 2 + A_2, A_2 = 1$$

$$\begin{aligned} \therefore a_n &= 2(3^n) + 1(n)(3^n) \\ &= 3^n(2 + n) \end{aligned}$$

That's it, we are done!

Let's quickly test for $n = 2$ and $n = 3$ to ensure we have the right solution. So using the recurrence,

$$\begin{aligned} a_2 &= 6a_{n-1} - 9a_{n-2} \\ &= 6a_{2-1} - 9a_{2-2} \\ &= 6a_1 - 9a_0 \\ &= 6(9) - 9(2) \\ &= 36, \end{aligned}$$

$$\begin{aligned} a_3 &= 6(36) - 9(9) \\ &= 135. \end{aligned}$$

By our explicit formula, we also obtain the same results:

$$a_2 = 3^2(2 + 2)$$

$$= 9(4)$$
$$= \boxed{36}, \checkmark$$

$$\begin{aligned} a_3 &= 3^3(2+3) \\ &= 27(5) \\ &= \boxed{135}. \checkmark \end{aligned}$$

†

Before moving on to the next type of problem, see if you can solve the explicit formula for geometric sequence from its recursive formula. Then, I encourage you to try the arithmetic sequence one as well.

Example 5.1.10

Find the explicit formula for each of these sequences:
Geometric Sequence:

$$a_n = \begin{cases} a_1 & \text{if } n = 1 \\ a_{n-1} \cdot r & \text{if } n > 1, \end{cases}$$

Arithmetic Sequence:

$$a_n = \begin{cases} a_1 & \text{if } n = 1 \\ a_{n-1} + d & \text{if } n > 1. \end{cases}$$

You should have obtained the following for the geometric sequence:

$$an = a_1(r)^{n-1}.$$

A number in an arithmetic sequence can also be derived through this explicit formula:

$$a_n = a_1 + d(n-1).$$

You may have had an easy time with the explicit formula for the geometric series, but how did manage to derive the explicit formula for the arithmetic sequence?

The arithmetic sequence is actually a non-homogeneous linear recurrence. Remember from Definition 5.1.2 that a homogeneous recurrence is one with only a_{n-k} terms for $k \in \mathbb{Z}^+$ ("k in positive integers set"). In this case, we have d as the constant, which does not have an a_{n-k} in the term, therefore it is non-homogeneous. Sequences in the form of

$$a_n = f(n)a_{n-1}$$

or

$$a_n = a_{n-1} + f(n),$$

where $f(n)$ is a non-constant function are considered non-homogeneous.

Solving non-homogeneous recurrences may be quite a challenging and complicated task. Let's break down how to solve non-homogeneous linear recurrences using the Characteristic Root Method. The other methods will be explained and demonstrated to solve both homogeneous and non-homogeneous linear recurrences after some brief examples.

Note that the Characteristic Root Method can be used only to solve for recurrences that take this form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are constants.

This can be manipulated to solve non-homogeneous recurrences in the form of

$$a_n = a_{n-1} + f(n)$$

but not

$$a_n = f(n) a_{n-1},$$

where $f(n)$ is a non-constant function (other methods that will be discussed in this chapter can also be used instead, such as the Iteration Method). This will be discussed in the following section.

Youtube Lectures 5.1

1. [Part 1] Solving Homogeneous Linear Recurrence Relations: Characteristic Root Method

2. [Part 2] Solving Homogeneous Linear Recurrence Relations: Characteristic Root Method

3. [Part 3] Solving Homogeneous Linear Recurrence Relations: Characteristic Root Method

Exercises 5.1

Exercise 5.1.1

$(a_n)_{m \geq 0}$ defines the sequence

$$3, 5, 11, 21, 43, 85, \ldots.$$

Write a recursive definition of a_n and then use the characteristic root method to find a closed formula for this sequence.

Exercise 5.1.2

Find a_n for the following values of n: $n = 3, 4, 5$. Then, solve the recurrence relation

$$a_0 = a_1 = 1, \quad a_2 = 2, \quad a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}.$$

(Note that $r^3 - 2r^2 - r + 2 = (r^2 - 1)(r - 2)$.)

Exercise 5.1.3

Suppose a linear homogeneous recurrence relation a_n has a characteristic equation with roots:

$$-3, -1, -1, -1, 3, 3, 4, 7, 9.$$

Find the form of a general solution for a_n.

Exercise 5.1.4

Given that $r^3 - 6r^2 + 12r - 8 = (r - 2)^3$ and $a_0 = 3$, $a_1 = 8$, $a_2 = 36$, $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$, solve the recurrence for a_n.

Exercise 5.1.5

Show that 4^n is a solution to the following recurrence relation:

$$a_n = 3a_{n-1} + 4a_{n-2}.$$

Exercise 5.1.6

Find the solution to the recurrence relation

$$a_n = -a_{n-1} + 6a_{n-2}$$

with initial terms $a_0 = 2$ and $a_1 = -1$.

Exercise 5.1.7

Given the initial terms $a_0 = 2$ and $a_1 = 5$, find the solution to the recurrence relation

$$a_n = 5a_{n-1} + 14a_{n-2}.$$

Then, find the solution to this recurrence if instead, $a_0 = 2$ and $a_1 = 11$.

Exercise 5.1.8

Given the recurrence relation

$$a_n = 5a_{n-1} + 6a_{n-2}$$

with initial terms $a_0 = 3$ and $a_1 = 13$, find the closed formula solution.

Exercise 5.1.9

Suppose that there are two solutions, r^n and q^n, of the recurrence relation

$$a_n = \alpha a_{n-1} + \beta a_{n-2}.$$

Prove that

$$c \cdot r^n + d \cdot q^n$$

is another solution to the a_n given any constants c, d.

5.2 Characteristic Root Method: Non-Homogeneous

But how do we solve non-homogeneous linear recurrences using this method?

Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(x),$$

where $f(x)$ denotes some non-constant function. Then, to solve these (separable) non-homogeneous recurrences, one may follow these three steps:

1. Find the homogeneous solution, denoted by $a_n^{(h)}$, which can be done by temporarily removing the extra non-homogeneous part from the sequence, $f(x)$, such that

$$a_n^{(h)} = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}.$$

2. Find the particular solution, denoted by $a_n^{(p)}$, by using a hypothesis of a_n based on $f(x)$.

   Here are some common $f(n)$'s you may see and a good hypothesis for each. If a hypothesis does not work, try a different one. Table 5.2.1 shows a list of hypothesis for each $f(x)$ (let d, e, A, A_x be constants; note that these are not strict hypotheses, meaning that if they do not work, you may try the next best option):

$f(n)$	Particular Solution Hypothesis $p(n)$
d	A
dn	$A_0 + A_1 n$
dn^2	$A_0+A_1n+A_2n^2$
ed^n	Ad^n

Table 5.2.1: Functions and Particular Solution Hypothesis

(There are many more that you may explore like sine, cosine, etc.)

3. Combine steps 1 and 2 to obtain the full solution:

$$a_n = a_n^{(h)} + a_n^{(p)},$$

which should have a constant left. You may use a given initial condition to solve for it.

As always, it is easier with an example, so let's get straight into a simple example.

Example 5.2.1

Let a_n be the arithmetic sequence,

$$a_n = a_{n-1} + 3, a_1 = 1,$$

where $n \geq 1$. Find an explicit formula for a_n.

Solution. Step 1: The homogeneous solution can be obtained by looking at $a_n = a_{n-1}$. From this, we can use our lambda hypothesis for $a_n - a_{n-1} = 0$:

$$\lambda^{n-1}(\lambda - 1) = 0,$$
$$\lambda = 1,$$

$$\begin{aligned} a_n^{(h)} &= A(1^n) \\ &= A. \end{aligned}$$

Step 2: The particular solution hypothesis would be based on $f(x)$, which is 3 in this case. Looking at Table 5.2.1, our hypothesis would be A. Since we already used A to denote the constant for the homogeneous solution, we may use a different one. Let's use γ ("gamma" greek letter):

$$a_n^{(p)} = \gamma.$$

Then, we use our hypothesis to substitute into the main equation and we get

$$\gamma = \gamma + 3.$$

Uh-oh, we cannot get a true particular solution, since $0 \neq 3$. That means this hypothesis did not work out, and it is because our coefficient on the a_{n-1} was the same as a_n. That is totally okay, we can now use a different one. Let's try the next closest thing, the linear hypothesis,

$$A_0 + A_1 n.$$

Following this obtains:

$$a_n^{(p)} = A_0 + A_1 n.$$

Substituting a_n for $a_n^{(p)}$, we get

$$A_0 + A_1 n = A_0 + A_1(n-1) + 3,$$

$$A_0 + A_1 n - A_0 - A_1 n + A_1 = 3,$$

$$A_1 = 3.$$

When $A_1 = 3$, any value of A_0 would make the equation true since it automatically canceled out. Therefore, A_0 does not contribute to the non-homogeneous term and there is no "requirement" placed by A_0.

It is very important to understand what a particular solution even does in this case. A particular solution contains terms necessary for balancing the main equation, which is why we have substituted $a_n^{(p)}$ for a_n to solve for the A_0, A_1 that helps balance it. In this case, A_0 does not matter so we only have to use

$$a_n^{(p)} = A_1 n$$

as our particular solution. Thinking of this logically too would make sense. $3n$ captures the only change happening to each term per increase in n, which is the non-homogeneous part of the sequence: adding 3 each time.

<u>Step 3</u>: Combine the two solutions to find the final solution.

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= A + 3n. \end{aligned}$$

We know $a_1 = 1$, so we can plug that into the equation to solve for A and obtain our final answer:

$$1 = A + 3, A = -2,$$

$$\therefore \boxed{a_n = 3n - 2}.$$

†

Notice that this is also equal to the traditional formula,

$$
\begin{aligned}
a_n &= a_1 + d(n-1) \\
&= 1 + 3(n-1) \\
&= 1 + 3n - 3 \\
&= 3n - 2.
\end{aligned}
$$

Pretty cool, right? Let's try another one.

Example 5.2.2

Let a_n be defined as the following recursive sequence:

$$a_n = 3a_{n-1} + 2n, a_1 = 3,$$

where $n \geq 1$. Formulate an explicit expression for this sequence.

Solution. Step 1: We obtain the homogeneous solution by getting rid of the $2n$ non-homogeneous part.

$$
\begin{gathered}
a_n = 3a_{n-1}, \\
a_n - 3a_{n-1} = 0, \\
\lambda^n - 3\lambda^{n-1} = 0, \\
\lambda^{n-1}(\lambda - 3) = 0 \\
\lambda = 3.
\end{gathered}
$$

So,

$$a_n^{(h)} = \alpha(3^n).$$

Step 2: Obtain the particular solution by using a hypothesis based on $2n$.
Looking at Table 5.2.1, we can use the linear hypothesis:

$$a_n^{(p)} = A_0 + A_1 n.$$

Substituting this into the main recurrence equation, we get:

$$
\begin{gathered}
a_n = 3a_{n-1} + 2n, \\
a_n - 3a_{n-1} - 2n = 0, \\
A_0 + A_1 n - 3(A_0 + A_1(n-1)) - 2n = 0, \\
A_0 + A_1 n - 3A_0 - 3A_1 n + 3A_1 - 2n = 0, \\
-2A_0 - 2A_1 n + 3A_1 - 2n = 0, \\
-2n(A_1 + 1) + (3A_1 - 2A_0) = 0.
\end{gathered}
$$

We understand that n is an arbitrary positive integer and that A_0, A_1 are constants. In order for this last equation to be true, under those constraints I just mentioned, A_1+1 must be 0; Otherwise, nothing else can control what n is and it will make the equation unbalanced. For example, if A_1+1 is some constant other than 0, then changing n will always change the value on the left side of the equation, as the rest of the equation is also constant.

But, we want the equation to be true in order to evaluate our constants and our particular solution. So we need the equation to always equal to 0, no matter the value of n. We can take advantage of the property of 0 that 0 times anything is 0 in order to expel the power of n, by setting

$$A_1 + 1 = 0.$$

Therefore, we must have A_1+1 equal to 0, which also means that the other constant expression $3A_1 - 2A_0$ must also equal 0:

$$A_1 + 1 = 0, A_1 = -1,$$

$$3A_1 - 2A_0 = 0,$$

$$\begin{aligned} A_0 &= (3/2)A_1 \\ &= (3/2)(-1) \\ &= -3/2. \end{aligned}$$

Thus,

$$\begin{aligned} a_n^{(p)} &= A_0 + A_1 n \\ &= -3/2 - n. \end{aligned}$$

Step 3: Obtain the final solution by combining the particular and homogeneous solutions.

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= \alpha(3^n) - 3/2 - n. \end{aligned}$$

Plugging in $a_1 = 3$,

$$3 = \alpha(3^(1)) - 3/2 - (1),$$

$$3 + 3/2 + 1 = 3\alpha,$$

$$11/6 = \alpha.$$

$$\therefore \boxed{a_n = (11/6)3^n - 3/2 - n}.$$

†

Let's test a_2 using the recurrence:

$$\begin{aligned} a_2 &= 3a_{n-1} + 2n \\ &= 3a_1 + 4 \\ &= 3(3) + 4 && \text{(Applying } a_1 = 3\text{)} \\ &= 13. \end{aligned}$$

The explicit formula we obtained also yields the same value for a_2:

$$\begin{aligned} a_n &= (11/6)3^n - 3/2 - n \\ &= (11/6)3^2 - 3/2 - 2 && \text{(Plugging in } n = 2\text{)} \\ &= (33/2) - 3/2 - 4/2 \\ &= 26/2 \\ &= \boxed{13}. \checkmark \end{aligned}$$

I'm sure you get the gist of this now. More problems using the other forms like exponential ones, which uses the same concepts and steps, is given below in Exercises 5.2.

Youtube Lectures 5.2

1. [Part 1] Solving Linear Non-Homogeneous Recurrence Relations: Characteristic Root Method
2. [Part 2] Solving Linear Non-Homogeneous Recurrence Relations: Characteristic Root Method

Exercises 5.2

Exercise 5.2.1

Consider the sequence

$$a_n = 2, 5, 12, 29, 70, 169, 408, \ldots.$$

If $a_0 = 2$, find a recursive definition and a closed formula for this sequence.

Exercise 5.2.2

Solve the recurrence relation $a_n = a_{n-1} + 2^n$ given that $a_0 = 2$.

Exercise 5.2.3

Consider the recurrence relation:

$$a_n = 3a_{n-1} + 2n + 1 \quad \text{for } n \geq 1,$$

with initial condition $a_0 = 5$. Use the Characteristic Root method to solve this recurrence. Feel free to refer back to Table 5.2.1.

Exercise 5.2.4

Let b_n be defined by the recurrence:

$$b_n = 4b_{n-1} - b_{n-2} + 6 \quad \text{for } n \geq 2,$$

with initial conditions $b_0 = 1$ and $b_1 = 2$. Solve this recurrence using the Characteristic Root technique.

Exercise 5.2.5

Define the sequence c_n by the relation:

$$c_n = 2c_{n-1} + 3c_{n-2} + 5^n \quad \text{for } n \geq 2,$$

with $c_0 = 0$ and $c_1 = 1$. Obtain a closed formula for this recurrence c_n.

Exercise 5.2.6

Consider the recurrence relation for the sequence d_n:

$$d_n = d_{n-1} + 2d_{n-2} + 7n \quad \text{for } n \geq 2,$$

with initial conditions $d_0 = 0$ and $d_1 = 1$. Find the explicit formula for this relation.

Exercise 5.2.7

Define e_n by the recurrence:

$$e_n = 5e_{n-1} + 3e_{n-2} + n^2 \quad \text{for } n \geq 2,$$

with initial conditions $e_0 = 2$ and $e_1 = 3$. Find the first few terms e_1, e_2, e_3. Then, solve the recurrence using the Characteristic Root technique.

5.3 Generating Functions Method

Remember generating functions? They are a useful method in modeling combinatorics problems, as explored in the previous chapter. Amazingly, generating functions can also be used to solve linear recurrence problems—here is an example of an interesting solution for a linear homogeneous recurrence using OGFs.

Example 5.3.1

The Fibonacci sequence can be defined by the closed expression called Binet's formula:

$$f_n = \frac{\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}{\sqrt{5}}.$$

Obtain this formula by solving the Fibonacci recurrence as shown below:

$$f_n = f_{n-1} + f_{n-2}, \text{ where } f_0 = 0 \text{ and } f_1 = 1.$$

We can solve this linear homogeneous recurrence by using the Characteristic Root Method, but OGFs also work! We begin by applying a cool trick:

Solution. We start by denoting $F(x)$ as the OGF of f_n as always, and we move every f_n to one side, obtaining

$$f_n - f_{n-1} - f_{n-2}$$

as an expression. Then, replace every f_n with $F(x)$ and multiply by x for every recurrence (switching every f_{n-k} term with $x^k F(x)$):

$$F(x) - xF(x) - x^2 F(x).$$

Note that this is an application of linear independence. Now we find an equal expression by using the definition of an OGF as

$$\sum_{n=0}^{\infty} a_n x^n,$$

so we have

$$\sum_{n=0}^{\infty} f_n x^n - x \sum_{n=0}^{\infty} f_n x^n - x^2 \sum_{n=0}^{\infty} f_n x^n.$$

Then, we use a smart indexing trick to equate this expression:

$$\sum_{n=0}^{\infty} f_n x^n - \sum_{n=1}^{\infty} f_{n-1} x^n - \sum_{n=2}^{\infty} f_{n-2} x^n.$$

But why is this equal? Well, for a single x contributed, it is equal to the same summation but just x^{n+1}, which then we can say $n+1 = z$ and so we have $z - 1 = n$, and thus, we get:

$$\sum_{z-1=0}^{\infty} f_{z-1}x^z = \sum_{z=1}^{\infty} f_{z-1}x^z.$$

Therefore, we arrive at the previous expression. We do the same for x^2 to shift the index by 2 to transform the entire expression for $F(x) - xF(x) - x^2F(x)$:

$$\begin{aligned} F(x) - xF(x) - x^2F(x) &= \sum_{n=0}^{\infty} f_n x^n - \sum_{n=1}^{\infty} f_{n-1}x^n - \sum_{n=2}^{\infty} f_{n-2}x^n \\ &= f_0x^0 + f_1x^1 + f_0x^1 + \sum_{n=2}^{\infty} (f_n - f_{n-1} - f_{n-2})x^n. \end{aligned}$$

The second expression was obtained by taking out the first few terms of the first two summations to combine into one summation. This is where the trick plays a big role: we actually know what $f_n - f_{n-1} - f_{n-2}$ is equal to, which comes from the original equation

$$f_n = f_{n-1} + f_{n-2},$$

$$f_n - f_{n-1} - f_{n-2} = 0.$$

Thus, we have

$$\begin{aligned} F(x) - xF(x) - x^2F(x) &= f_0x^0 + f_1x^1 + f_0x^1 \\ &= f_0 + x(f_1 + f_0) \\ &= 0 + x(1 + 0) \qquad \text{(Plugging in given values for } f_0, f_1\text{)} \\ &= x. \end{aligned}$$

Therefore, this results in an equation in which we can solve for the generating function of this sequence $F(x)$:

$$\begin{aligned} F(x)(1 - x - x^2) &= x, \\ F(x) &= \frac{x}{1 - x - x^2}. \end{aligned}$$

There we have it, that is the OGF of the recurrence. Now how do we find an explicit formula for it? We need to find the term of x^n inside a sum that can give us the coefficient a_n (by the standard definition of generating functions).

So, we can express $F(x)$ as a partial fraction and do partial fraction decomposition in order to convert some geometric series formula into the power series form. But instead of factoring the

$$(1 - x - x^2)$$

into

$$(x - \alpha)(x - \beta),$$

it will be easier to immediately force it into a form of

$$(1 - \alpha x)(1 - \beta x),$$

since the geometric sum formula for

$$x^0 + x + x^2 + \cdots = 1 + x + x^2 + \dots$$

can be reverse-substituted for

$$\frac{1}{1-x}.$$

If it does not make sense now, it should in a bit—we are basically trying to simplify this expression into a sum using the geometric sum formula that

$$1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

The goal is to find the α and β in the expression in order to factor it. We set

$$\begin{aligned} 1 - x - x^2 &= (1-\alpha x)(1-\beta x) \\ &= 1 - \alpha x - \beta x + \alpha\beta x^2 \\ &= 1 - x(\alpha+\beta) - x^2(-\alpha\beta). \end{aligned}$$

Then, $\alpha + \beta$ must equal to 1 and $-\alpha\beta$ must equal to 1 as well:

$$(1)\ \alpha + \beta = 1,$$

$$(2)\ -\alpha\beta = 1.$$

From condition 1,

$$\beta = 1 - \alpha$$

and plugging into condition 2 to solve for α, we get:

$$\begin{aligned} -\alpha(1-\alpha) &= -\alpha + \alpha^2 \\ &= 1. \end{aligned}$$

Thus, arranging the equation obtains:

$$\alpha^2 - \alpha - 1 = 0.$$

The quadratic equation then gives a solution for α,

$$\begin{aligned} \alpha &= \frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{1 \pm \sqrt{5}}{2}. \end{aligned}$$

Condition 1 then allows us to solve for β as well by substitution:

$$\alpha + \beta = 1,$$
$$\beta = 1 - \alpha,$$

$$\begin{aligned}\beta &= 1 - \frac{1 \pm \sqrt{5}}{2}\\ &= \frac{2 - (1 \pm \sqrt{5})}{2}\\ &= \frac{1 \mp \sqrt{5}}{2}.\end{aligned}$$

The solution we get for α and β is the same no matter which sign we pick. Consequently, let

$$\alpha = \frac{1 + \sqrt{5}}{2}$$

and

$$\beta = \frac{1 - \sqrt{5}}{2}.$$

Then, we have the simplified the generating function expression:

$$\begin{aligned}F(x) &= \frac{x}{(1 - \alpha x)(1 - \beta x)}\\ &= \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}.\end{aligned}$$

To solve for A and B in the partial fraction decomposition, we multiply everything by the denominator of the left side and get:

$$x = A(1 - \beta x) + B(1 - \alpha x).$$

We plug in

$$x = \frac{1}{\beta}$$

to isolate an equation for B (those values for x will make the term with A on the right side become 0):

$$\begin{aligned}\frac{1}{\beta} &= A(1 - 1) + B(1 - \frac{\alpha}{\beta}) && \text{(Letting } x = \tfrac{1}{\beta}\text{)}\\ &= B(1 - \frac{\alpha}{\beta}).\end{aligned}$$

Divide both sides by $1 - \frac{\alpha}{\beta})$ to solve for B:

$$\begin{aligned}
B &= \frac{1}{\beta(1-\frac{\alpha}{\beta})} \\
&= \frac{1}{\beta-\alpha} \\
&= \frac{1}{\frac{1-\sqrt{5}}{2}-\frac{1+\sqrt{5}}{2}} \\
&= \frac{1}{\frac{1-\sqrt{5}-1-\sqrt{5}}{2}} \\
&= \frac{1}{\frac{-2\sqrt{5}}{2}} \\
&= -\frac{1}{\sqrt{5}}.
\end{aligned}$$

To solve for A, we can set $x = 0$:

$$\begin{aligned}
x &= A(1-\beta x) + B(1-\alpha x), \\
0 &= A + B, \\
A &= -B \\
A &= \frac{1}{\sqrt{5}}.
\end{aligned}$$

Plugging these two values back into the equation we obtained through partial fraction decomposition originally, we have an equation for the OGF $F(x)$:

$$\begin{aligned}
F(x) &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \\
&= \frac{\frac{1}{\sqrt{5}}}{1-\alpha x} + \frac{-\frac{1}{\sqrt{5}}}{1-\beta x}.
\end{aligned}$$

Then we apply the geometric sum formula

$$\begin{aligned}
\sum_{k=0}^{\infty} x^n &= 1 + x + x^2 + \dots \\
&= \frac{1}{1-x},
\end{aligned}$$

where x is replaced by $\alpha x, \beta x$. This results in the following equation:

$$F(x) = \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\alpha x)^n - \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\beta x)^n$$

$$= \frac{1}{\sqrt{5}}(\sum_{k=0}^{\infty} \alpha^n x^n - \sum_{k=0}^{\infty} \beta^n x^n)$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{\infty} (\alpha^n - \beta^n) x^n. \qquad \text{(Linearity of Sums)}$$

$\therefore$ the coefficient of x^n is precisely

$$\frac{1}{\sqrt{5}}(\alpha^n - \beta^n) = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

Plugging in our original values that we set for α, β, we finally get Binet's formula:

$$\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2},$$

$$f_n = \boxed{\frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}}.$$

†

This is a spectacular proof of Binet's explicit formula for the Fibonacci sequence. Now, let's explore a non-homogeneous example:

Example 5.3.2

Let

$$a_n = 2a_{n-1} + 3$$

for $a \geq 0$. Given that $a_0 = 1$, solve an explicit formula for a_n using OGFs.

Solution. First, move everything to one side:

$$a_n - 2a_{n-1} - 3 = 0.$$

For the OGF trick, we use the form on the left side and for the sake of linear independence, we replace every a_{n-k} term with $x^k A(x)$, where $A(x)$ is the OGF of this recurrence function a_n:

$$a_n - 2a_{n-1} - 3 = 0 \to A(x) - 2xA(x) - 3.$$

Next, we also switch -3 with its OGF (recall that all OGFs take the form $\sum_{n=0}^{\infty} p_n x^n$, where p_n is the function):

$$A(x) - 2xA(x) - \sum_{n=0}^{\infty}(3)x^n = A(x) - 2xA(x) - 3\sum_{n=0}^{\infty} x^n$$

$$=?$$

Now, we replace the $A(x)$ with its definition as an OGF of a_n to obtain the second expression "?" (note that our goal is to equate the $A(x)$ expression to another expression "?" in order to create an equation to solve for $A(x)$):

$$\sum_{n=0}^{\infty} a_n x^n - 2x \sum_{n=0}^{\infty} a_n x^n - 3 \sum_{n=0}^{\infty} x^n.$$

Next, distribute the x of the second term to inside the summation:

$$\sum_{n=0}^{\infty} a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^{n+1} - 3 \sum_{n=0}^{\infty} x^n.$$

Use index shifting of $n+1=z$ for the second term and switch it back to n:

$$\begin{aligned} -2 \sum_{z-1=0}^{\infty} a_{z-1} x^z &= -2 \sum_{z=1}^{\infty} a_{z-1} x^z \\ &= -2 \sum_{n=1}^{\infty} a_{n-1} x^n. \end{aligned}$$

Bringing it all together, the entire expression simplifies to

$$\sum_{n=0}^{\infty} a_n x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n - 3 \sum_{n=0}^{\infty} x^n.$$

We can now turn every summation into the same starting index in order to put it all as one sum like so, by taking out the $n=0$ expressions of the first and third terms:

$$a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n - 3(x^0 + \sum_{n=1}^{\infty} x^n)$$

x^0 is just 1 and $a_0 = 1$ was given in the problem. So we get constants of $1 - 3 = -2$ added by the rest:

$$-2 + \sum_{n=1}^{\infty} a_n x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n - 3 \sum_{n=1}^{\infty} x^n.$$

(Now combine the sums since they now share the same indexing)

$$\begin{aligned} &= -2 + \sum_{n=1}^{\infty} a_n x^n - 2a_{n-1} x^n - 3x^n \\ &= -2 + \sum_{n=1}^{\infty} x^n (a_n - 2a_{n-1} - 3). \end{aligned}$$

But notice that we already know that

$$a_n - 2a_{n-1} - 3 = 0$$

from the initial manipulation when we moved everything to one side. Therefore, this expression is just equal to

$$-2+\sum_{n=1}^{\infty}x^n(a_n-2a_{n-1}-3)=-2+\sum_{n=1}^{\infty}x^n(0). \qquad =-2.$$

So, if we bring back the original OGF expression that got us here, we obtain an equation:

$$A(x)-2xA(x)-3\sum_{n=0}^{\infty}x^n=-2,$$

$$A(x)(1-2x)=-2+3\sum_{n=0}^{\infty}x^n.$$

We know

$$\sum_{n=0}^{\infty}x^n$$

represents a geometric sum, which is equal to

$$\frac{1}{1-x}$$

by the geometric sum formula. Thus, we have:

$$\begin{aligned}A(x)(1-2x)&=-2+\frac{3}{1-x}\\&=\frac{-2+2x+3}{1-x}.\end{aligned}$$

Solving this equation for the OGF $A(x)$,

$$A(x)=\frac{1+2x}{(1-x)(1-2x)}.$$

Finally, remember that the goal of finding the OGF, $A(x)$, of this sequence a_n is to obtain the coefficient of x_n within $A(x)$, which would reveal the explicit formula for a_n by definition of generating functions. So we need to express this $A(x)$ value into a sum with a term coefficient of x^n, which we can do by first using partial fraction decomposition and then using the geometric sum formula to turn it into sums that we can combine to find the coefficient of x^n.

Using partial fraction decomposition:

$$\begin{aligned}A(x)&=\frac{1+2x}{(1-x)(1-2x)}\\&=\frac{A}{1-x}+\frac{B}{1-2x}.\end{aligned}$$

Thus, by multiplying $(1-x)(1-2x)$ on both sides,

$$1+2x=A(1-2x)+B(1-x).$$

Plug in $x = 1$ to obtain the value for A:

$$\begin{aligned} 1 + 2x &= A(1 - 2x) + B(1 - x), \\ 1 + 2(1) &= A(-1) + B(0), && \text{(Set } x = 1\text{)} \\ 3 &= -A. \end{aligned}$$

Therefore,

$$A = -3.$$

Now, we set $x = \frac{1}{2}$ to solve for B:

$$\begin{aligned} 1 + 2x &= A(1 - 2x) + B(1 - x), \\ 1 + \frac{1}{2}(2) &= A(1 - \frac{1}{2}(2)) + B(1 - \frac{1}{2}), \\ 2 &= A(0) + B(\frac{1}{2}), \\ 2 &= \frac{B}{2}, \\ B &= 4. \end{aligned}$$

So,

$$A(x) = \frac{-3}{1 - x} + \frac{4}{1 - 2x}.$$

We can again use the infinite geometric sum formula

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n,$$

where we get the following (for the second expression, x is replaced by $2x$):

$$-3\frac{1}{1 - x} = -3\sum_{n=0}^{\infty} x^n,$$

$$4\frac{1}{1 - 2x} = 4\sum_{n=0}^{\infty} (2x)^n.$$

Substituting in the new, equivalent terms,

$$\begin{aligned} A(x) &= -3\sum_{n=0}^{\infty} x^n + 4\sum_{n=0}^{\infty} (2x)^n \\ &= \sum_{n=0}^{\infty} -3x^n + \sum_{n=0}^{\infty} (4)(2^n)x^n. \end{aligned}$$

We can combine all the sums since they have all the same starting and ending indices:

$$A(x) = \sum_{n=0}^{\infty} [-3x^n + (4)(2^n)x^n].$$

And finally, we can express this as a single sum with a coefficient of x^n through the distributive property:

$$A(x) = \sum_{n=0}^{\infty} x^n [-3 + (4)(2^n)].$$

The coefficient of x^n in the sum given by the generating function is a_n, by definition.

$$\begin{aligned} \therefore a_n &= -3 + (4)(2^n) \\ &= \boxed{2^{n+2} - 3}. \end{aligned}$$

†

Wow! That was quite a lengthy process but this is another tool you may consider when looking to solve non-homogeneous linear recurrences—the more you practice, the easier it will get. This approach for non-homogeneous recurrence relations is similar to the approach for homogeneous, but just adding the OGF for the extra $f(x)$ (the non-homogeneous part).

Feel free to also check the answer by using the Characteristic Root Method, you should be able to get the same answer.

Youtube Lectures 5.3

1. [Part 1] Using Generating Functions to Solve Non-Homogeneous Recurrence Relations

2. [Part 2] Using Generating Functions to Solve Non-Homogeneous Recurrence Relations

Exercises 5.3

Exercise 5.3.1

Suppose $T_0 = 2$, $T_1 = 3$, and $T_2 = 6$. For $n \geq 3$, let T_n be defined by the recurrence $T_n = -2T_{n-1} + 5T_{n-2} + 6T_{n-3}$. If

$$r^3 + 4r^2 - 5r - 6 = (r+1)(r-2)(r+3),$$

then find a closed formula for this recurrence relation using generating functions.

Exercise 5.3.2

Find a_n for $n = 2, 3, 4$ given the recurrence relation:

$$a_0 = a_1 = 1, \quad a_n = 3a_{n-1} + 4a_{n-2}.$$

Then, solve it using a generating functions approach.

Exercise 5.3.3

A population of rabbits grows in a way such that each year's population depends on the previous two years. Each year's population is three times the population of the previous year minus twice the population two years ago. If there were 2 rabbits initially and 6 rabbits during the first year, find the population after n years using a generating functions approach:

$$r_n = 3r_{n-1} - 2r_{n-2}, \quad r_0 = 2,\ r_1 = 6.$$

Exercise 5.3.4

Suppose

$$a_n = 2a_{n-1} - a_{n-2}, \quad a_0 = 3000,\ a_1 = 1000.$$

Find the generating function for a_n and use that to determine an explicit function for a_n.

Exercise 5.3.5

The weather fluctuates with temperature t_n in the AWM national park, as modeled by the recursive function below:

$$t_n = 4t_{n-1} - 4t_{n-2}, \quad t_0 = 1,\ t_1 = 4.$$

Solve this recursion using generating functions.

Exercise 5.3.6

Suppose the sequence $\{a_n\}$ is defined by the following: $a_0 = 0$, $a_1 = 1$, and $a_n = 3a_{n-1} - 2a_{n-2}$.

1. Let $A(x) = a_0 + a_1x + a_2x^2 + \dots$ be the generating function for a_n. Use this to prove that
$$A(x) = \frac{x}{1-3x+2x^2}.$$
2. Given
$$\frac{x}{1-3x+2x^2} = \frac{1}{1-2x} - \frac{1}{1-x},$$
find a closed-form expression for a_n.

Exercise 5.3.7

Let $\{b_n\}$ be a sequence where $b_0 = 2$ and $b_1 = 9$. If $b_n = 6b_{n-1} - 9b_{n-2}$,

1. find a closed form for the generating function
$$B(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots,$$
2. find an explicit expression for b_n using the Binomial Theorem.

Exercise 5.3.8

Find a closed-form expression for the generating function of the sequence $c_n = 3^n$ (for $n \geq 0$).

Exercise 5.3.9

Find a closed-form expression for the generating function of the sequence d_n, where
$$d_n = \begin{cases} 3^n & \text{for } n \text{ even,} \\ 2 \cdot 3^n & \text{for } n \text{ odd.} \end{cases}$$

5.4 Proof by Induction Method

Are you good at identifying patterns? If so, then that is great–you'll need that for this method. Let me give you some example problems to test that skill real quick.

Example 5.4.1

Identify the pattern for each of these three cases:

1. $a_n = 1, 2, 4, 8, \ldots$
2. $b_n = 5, 8, 12, 16, 16, 0, -64, \ldots$ (hint: find its pattern with 2^n sequence: $1, 2, 4, 8, 16, \ldots$)
3. $c_n = 1, 3, 7, 15, 31, 63, \ldots$

Now, write each of them in a closed-formula (explicit) form, like $a_n = 5n$ for example, with the first n being $n = 1$ ($n \geq 1$).

Generally, for proof by induction, there are two steps that you want to prove: The base case and the inductive step.

Let's take a look at a basic non-recurrence example to show how each of them works and then we will explore its applications for linear recurrence relations. (The solutions to the example above will be revealed after this example.)

Example 5.4.2

Say we want to prove the sum of the first n positive integers is equal to the well-known equation,

$$s(n) = \frac{n(n+1)}{2}.$$

Solution. We start with proving the base case. The base case is the first case you want to prove to be true. But why do we want to do this in the first place?

We are basically trying to prove that the formula that we proposed holds true for all domains of $s(n)$, in other words,

$$\forall n \in \mathbb{Z}^+$$

("for all n in the set of positive integers", which is essentially $n = 1, 2, 3, \ldots$). We want to prove that the first domain of $s(n)$ is true, in this case, it is $n = 1$. Then, in the inductive step, we will prove that the formula works for $s(k+1)$ if we know that $s(k)$ holds true for that formula too.

If we are able to reach these two conclusions, then we can say that it works for the base case, at $n = 1$, meaning the formula will work for $n = 2$ by the inductive step, and for $n = 3$, and so on, proving our hypothesis correct.

Therefore, in this example, we want to test out the formula if it works for $n = 1$. We know that the sum of the first 1 positive integer is just equal to 1:

Base Case ($n = 1$):

$$s(1) = \frac{1(1+1)}{2}$$

$$= \frac{(2)}{2}$$
$$= 1. \checkmark$$

Now that we have proven the base case true, we need to prove the inductive step.

In the inductive step, we prove the case true for $s(k+1)$ by assuming $s(k)$. So, assuming

$$s(k) = \frac{k(k+1)}{2},$$

we use the formula for when $k = k+1$:

$$\begin{aligned} s(k+1) &= \frac{(k+1)((k+1)+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

And also by default, we know

$$s(k) = 1 + 2 + 3 + \cdots + k.$$

And that

$$s(k+1) = \underbrace{1 + 2 + 3 + \cdots + k}_{s(k)} + (k+1),$$

which means

$$s(k+1) = s(k) + (k+1).$$

Using our formula to substitute for $s(k)$ and $s(k+1)$, we get:

$$\frac{(k+1)(k+2)}{2} = \frac{k(k+1)}{2} + k + 1$$

So now if the equation is equal on both sides, that means the formula works for $s(k+1)$. Why? Well, you can think of this as plugging in values for x as a hypothesis into $3x - 4 = 5$, and if the left side was equal to the right side, then we know that the value for x is true for that equation. This is the same thing here, we are using the formula for $s(k)$ as a hypothesis and plugging into the equation for $s(k+1)$ that we already knew was true.

Simplifying, the right side becomes

$$\begin{aligned} \frac{(k+1)(k+2)}{2} &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k(k+1)}{2} + \frac{2k+2}{2} \\ &= \frac{k(k+1) + 2k + 2}{2} \end{aligned}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}. \checkmark$$

The right side is indeed equal to the left side, and therefore, we have proved the equation

$$\frac{n(n+1)}{2}$$

true for $s(n)$, the sum of the first k positive integers. †

Before we jump into how to use this to solve recurrences, the answers to the three questions in the beginning are:

Solution. Purely based on pattern recognition, the following can be obtained:

$$a_n = 2^{n-1},$$

$$b_n = (5-n)2^n,$$

$$c_n = 2^n - 1.$$

†

Great! Below is a few steps you may follow to apply this proof technique for recurrences:

Definition 5.4.3 (Proof by Induction Steps for Recurrences)
To solve linear recurrences, the following steps may be followed to prove an explicit formula definition of the sequence:

1. Test the first few values using the recurrence formula and our given conditions,
2. find the pattern and formulate an explicit formula for it,
3. test the base case, and
4. prove our formula true by using proof by induction.

Just a quick note, the formula that we assume for any proof by induction method is called the induction hypothesis.

Definition 5.4.4 (Proof by Induction)
Given a sequence that can be defined through a recurrence relation, an initial induction hypothesis may be formed (as a closed formula). A proof by induction assumes this induction hypothesis of n and proves it correct for the base case(s), and then tests if this hypothesis is true for a step $n+1$, which will prove the case after the base case correct, the case after that, etc.—in other words, proving this step shows that all integers greater than the base case would be correct.

Let's look at a second-order linear homogeneous recurrence example.

Example 5.4.5
Let n be a non-negative integer ($n \geq 1$). Given the recurrence relation a_n where

$$a_n = 2a_{n-1} - a_{n-2}$$

and $a_0 = 0, a_1 = 3$, find a closed formula for this sequence.

Solution. Following the first step from Definition 5.4.3, we test the first few values using our given values:

$$\begin{aligned} a_2 &= 2a_1 - a_0 \\ &= 2(3) - 0 && \text{(substituting known values } a_0, a_1\text{)} \\ &= 6, \end{aligned}$$

$$\begin{aligned} a_3 &= 2a_2 - a_1 \\ &= 2(6) - 3 && \text{(substituting known values } a_1, a_2\text{)} \\ &= 9, \end{aligned}$$

$$\begin{aligned} a_4 &= 2a_3 - a_2 \\ &= 2(9) - 6 && \text{(substituting known values } a_2, a_3\text{)} \\ &= 12, \end{aligned}$$

$$\begin{aligned} a_5 &= 2a_4 - a_3 \\ &= 2(12) - 9 && \text{(substituting known values } a_3, a_4\text{)} \\ &= 15, \end{aligned}$$

$$\begin{aligned}
a_6 &= 2a_5 - a_4 \\
&= 2(15) - 12 && \text{(substituting known values } a_4, a_5\text{)} \\
&= 18,
\end{aligned}$$

$$\vdots$$

Following the second step of Definition 5.4.3, we clearly see a pattern of $6, 9, 12, 15, 18, \ldots$ for $n = 2, 3, 4, 5, 6, \ldots$:

$$a_n = 3n.$$

Following the third step of Definition 5.4.3, we then test the base case with $k = 0$. We clearly see that

$$\begin{aligned}
a_0 &= 3n \\
&= 3(0) \\
&= 0,
\end{aligned}$$

proving that our formula also works when $k = 0$. This is also true for $a_1 = 3$, as we have

$$\begin{aligned}
a_0 &= 3n \\
&= 3(1) \\
&= 3.
\end{aligned}$$

In the fourth step of Definition 5.4.3, we perform our inductive step, where we test if the formula for a_k works when $k = k + 1$. Our assumption is:

$$a_k = 3k.$$

(Given) Recurrence case for $k + 1$:

$$a_{k+1} = 2a_k - a_{k-1}.$$

Plugging into the equation above for both sides using assumption, we obtain

$$3(k + 1) = 2(3(k)) - 3(k - 1).$$

The right side can be simplified to:

$$
\begin{aligned}
2(3(k)) - 3(k-1) &= 6(k) - 3(k-1) \\
&= 6k - 3k + 3 \\
&= 3k + 3 \\
&= 3(k+1). \checkmark
\end{aligned}
$$

The right side is equal to the left side, so we are finished.
$\therefore$, our induction hypothesis as the explicit formula

$$a_n = 3n$$

holds true under the recurrence

$$\forall n \geq 0 \text{ s.t. } n \in \mathbb{Z}.$$

†

This was an easy example. Let's now try a harder linear non-homogeneous first-order recurrence:

Example 5.4.6

Let $h_0 = 4$ and n is a non-negative integer ($n \geq 1, n \in \mathbb{Z}^+$). Then, the sequence h_n is defined as

$$h_n = 2h_{n-1} + 3^n.$$

Find an explicit formula for this recurrence relation.

Solution. Begin by finding the first few terms:

$$h_0 = 4,$$

$$
\begin{aligned}
h_1 &= 2h_0 + 3^1 \\
&= 2(4) + 3 && \text{(substituting known values } h_0\text{)} \\
&= 11,
\end{aligned}
$$

$$
\begin{aligned}
h_2 &= 2h_1 + 3^2 \\
&= 2(11) + 9 && \text{(substituting known values } h_1\text{)} \\
&= 31,
\end{aligned}
$$

$$h_3 = 2h_2 + 3^3$$

$$
\begin{aligned}
&= 2(31) + 27 && \text{(substituting known values } h_2\text{)} \\
&= 89,
\end{aligned}
$$

$$
\begin{aligned}
h_4 &= 2h_3 + 3^4 \\
&= 2(89) + 81 && \text{(substituting known values } h_3\text{)} \\
&= 259,
\end{aligned}
$$

$$\vdots$$

This may be quite difficult to spot at first, but if you realize how fast it is growing, it kind of seems like it is growing by powers of 3 starting at 31 when $n = 0$, since powers of 3 go by $3, 9, 27, 81, 243, \ldots$ (let b_n be a temporary/side function):

$$b_n = 3^{n+1}.$$

If we subtract off the powers of 3 from each recurrence that we checked, we reveal a different pattern (let c_n also be a temporary/side function):

$$(c_n) = 1, 2, 4, 8, 16$$

$$\to c_n = 2^n.$$

Thus, an induction hypothesis can be formed:

$$
\begin{aligned}
h_n &= b_n + c_n \\
&= 3^{n+1} + 2^n.
\end{aligned}
$$

Next, we check the base case for $n = 0$ for using the formula, and we get that indeed,

$$
\begin{aligned}
h_0 = 3^{n+1} + 2^n \\
&= 3^{0+1} + 2^0 \\
&= 3 + 1 \\
&= 4.
\end{aligned}
$$

This is correct by our initial given condition.
All left now is the induction step:
Assumption:

$$h_k = 3^{k+1} + 2^k.$$

(Given) Recurrence Case for $k+1$:

$$h_{k+1} = 2h_k + 3^{k+1}.$$

Substituting into both sides of the equation above using our assumption:

$$3^{k+2} + 2^{k+1} = 2(3^{k+1} + 2^k) + 3^{k+1}$$

The right side can be simplified to:

$$\begin{aligned} 2(3^{k+1} + 2^k) + 3^{k+1} &= 2(3^{k+1}) + 2(2^k) + 3^{k+1} \\ &= 3(3^{k+1}) + 2^{k+1} \\ &= 3^{k+2} + 2^{k+1}. \checkmark \end{aligned}$$

Since the right side equals the left, we are done.
Therefore, our assumption of

$$h_k = 3^{k+1} + 2^k$$

for this particular recurrence is proven,

$$\forall n \geq 0 \text{ s.t. } n \in \mathbb{Z}.$$

†

Finding these patterns may be quite challenging, which is why we introduce the next two methods that break down the pattern for these sequences.

Exercises 5.4

Exercise 5.4.1

Let $a_1 = 1$ and $a_2 = 2$. If $a_n = 3a_{n-1} - 2a_{n-2}$ $\forall n \geq 3$, then find a closed formula for a_n using proof by induction.

Exercise 5.4.2

Find a pattern in the sequence and conjecture a closed formula for the amount in the account after n months, given by the formula below:

$$a_n = 6a_{n-1} - 9a_{n-2}, \quad a_0 = 2, \, a_1 = 9.$$

Then, prove through induction.

Exercise 5.4.3

Back in Exercise 5.1.8, the recurrence relation

$$a_n = 5a_{n-1} + 6a_{n-2}$$

with initial terms $a_0 = 3$ and $a_1 = 13$ was solved. Prove this closed-form formula true through proof by induction.

Exercise 5.4.4

The non-homogeneous linear recurrence relation $a_n = a_{n-1} + 2^n$ given that $a_0 = 2$ was solved in Exercise 5.1.8. Prove this formula through induction.

Exercise 5.4.5

Exercise 5.1.6 explored the solution to the recurrence relation $a_n = -a_{n-1} + 6a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = -1$. Use induction to prove that this formula holds.

5.5 Iteration/Back-Substitution Method

Definition 5.5.1 (Back-Substitution Method)

Back-substitution for solving linear recurrence relation is a method that determines the value of explicit terms by recursively substituting using equations created by known initial conditions. This process is repeated until a clear pattern is found.

So this method is quite easy, although it can take a bit of substituting. It is generally easier for first-order recurrences, so we will solve these example linear recurrences: first, a first-order non-homogeneous recurrence, then a second-order homogeneous recurrence.

Example 5.5.2

A linear non-homogeneous recurrence a_n is given by

$$a_n = 2a_{n-1} - 2^n.$$

If $a_0 = 5$ and $n \geq 1$, then find a closed formula for a_n.

Solution. Let's start the iterating process right away:

$$a_n = 2a_{n-1} - 2^n.$$

From the recurrence, we know that

$$a_{n-1} = 2a_{n-2} - 2^{n-1}.$$

So we plug it in the original equation for a_n, hence the name "back-substitution":

$$\begin{aligned} a_n &= 2(2a_{n-2} - 2^{n-1}) - 2^n && \text{(substituting for } a_{n-1}\text{)} \\ &= (2^2)a_{n-2} - (2)(2^{n-1}) - 2^n \\ &= (2^2)a_{n-2} - 2^n - 2^n. \end{aligned}$$

Now we keep doing this, now with our new equation, until we notice a pattern:
Given

$$a_{n-2} = 2a_{n-3} - 2^{n-2},$$

$$\begin{aligned} a_n &= (2^2)(2a_{n-3} - 2^{n-2}) - 2^n - 2^n && \text{(substituting for } a_{n-2}\text{)} \\ &= (2^2)(2)a_{n-3} - (2^2)2^{n-2} - 2^n - 2^n \\ &= (2^3)a_{n-3} - 2^n - 2^n - 2^n. \end{aligned}$$

Again, we know the following based on the original recurrence relation:

$$a_{n-3} = 2a_{n-4} - 2^{n-3}.$$

Now, we again substitute this equation into the previously obtained one:

$$\begin{aligned} a_n &= (2^3)(2a_{n-4} - 2^{n-3}) - 2^n - 2^n - 2^n && \text{(substituting for } a_{n-3}\text{)} \\ &= (2^3)(2)a_{n-4} - (2^3)2^{n-3} - 2^n - 2^n - 2^n \\ &= (2^4)a_{n-4} - 2^n - 2^n - 2^n - 2^n. \end{aligned}$$

So, we have:

$$\begin{aligned} a_n &= 2a_{n-1} - 2^n && \text{(Iteration 1)} \\ a_n &= (2^2)a_{n-2} - 2^n - 2^n && \text{(Iteration 2)} \\ a_n &= (2^3)a_{n-3} - 2^n - 2^n - 2^n && \text{(Iteration 3)} \\ a_n &= (2^4)a_{n-4} - 2^n - 2^n - 2^n - 2^n. && \text{(Iteration 4)} \end{aligned}$$

Do you see the pattern now?
Generalizing this into an explicit formula for the kth iteration obtains:

$$a_n = 2^k a_{n-k} - (k)2^n.$$

This is not quite yet our solution, as we need a_n only in terms of n. Remember, we were given $a_0 = 5$. But how do we use it?

Well, we need to get rid of the a_{n-k} term along with the other k terms. So why not set a value for k such that we obtain a_0, by applying

$$n - k = 0.$$

If

$$n - k = 0, \text{ then } n = k.$$

Subsequently, we substitute every k for n, since we only want everything in terms of n:

$$\begin{aligned} a_n &= 2^k a_{n-k} - (k)2^n \\ &= 2^n a_{n-n} - (n)2^n && \text{(Substitute } k = n\text{)} \\ &= 2^n a_0 - (n)2^n \\ &= 2^n 5 - (n)2^n. \end{aligned}$$

Therefore,

$$\boxed{a_n = 2^n(5 - n)}.$$

†

This may seem like a lot of work for one problem but the more practice you get with it, the faster you will be able to see the pattern. It is all about manipulating the expressions you get per iteration.

Alright, now let's try a second-order linear homogeneous example with the iteration method:

Example 5.5.3

a_n is a recurrence relation as follows:

$$a_n = 3a_{n-1} + 4a_{n-2}.$$

Given that

$$a_0 = a_1 = 1,$$

find a closed formula for a_n.

Solution. We start by listing two expressions for

$$a_{n-1}, a_{n-2}$$

by using the main equation:

(replacing every n with $n - 1$)

$$a_{n-1} = 3a_{n-2} + 4a_{n-3},$$

(replacing every n with $n-2$)

$$a_{n-2} = 3a_{n-3} + 4a_{n-4}.$$

These equations we know must be true since we used the main equations. Now, we can plug these into the main equation:

$$a_n = 3a_{n-1} + 4a_{n-2} = 3(3a_{n-2} + 4a_{n-3}) + 4(3a_{n-3} + 4a_{n-4}) = 3^2 a_{n-2} + 3(4a_{n-3}) + 4(3a_{n-3}) + 4^2 a_{n-4} =$$

Notice that we do not simplify the $3^2, 4^2$, and $2(4)(3)$ because we want to spot a pattern in this method, so keeping it in those power and un-simplified forms will help us do that more easily.

Next, we perform the same thing for the new equation we got, which we will need to evaluate a_{n-3}, a_{n-4} first using our original $a_n = 3a_{n-1} + 4a_{n-2}$ equation:

(replacing every n with $n-3$)

$$a_{n-3} = 3a_{n-4} + 4a_{n-5},$$

(replacing every n with $n-4$)

$$a_{n-4} = 3a_{n-5} + 4a_{n-6}.$$

Let's plug $a_{n-2}, a_{n-3}, a_{n-4}$ into our new equation that we had obtained (a_{n-2} we already found an expression for earlier):

$$\begin{aligned}
a_n &= (3^2)a_{n-2} + (2)(4(3a_{n-3})) + (4^2)a_{n-4} \\
&= 3^2(3a_{n-3} + 4a_{n-4}) + 2(4(3(3a_{n-4} + 4a_{n-5}))) + 4^2(3a_{n-5} + 4a_{n-6}) \\
&= 3^3 a_{n-3} + 3^2(4)a_{n-4} + 2(4)(3^2)a_{n-4} + 2(4^2)(3)a_{n-5} + 4^2(3)a_{n-5} + 4^3 a_{n-6} \\
&= (3^3)a_{n-3} + (3)(4)(3^2)a_{n-4} + (3)(4^2)(3)a_{n-5} + (4^3)a_{n-6}.
\end{aligned}$$

We see some patterns, but it's not too clear. Let's do it one last time:

(replacing every n with n-5)

$$a_{n-5} = 3a_{n-6} + 4a_{n-7},$$

(replacing every n with n-6)

$$a_{n-6} = 3a_{n-7} + 4a_{n-8}.$$

Plugging in the four terms:

$$\begin{aligned}
a_n = {} & (3^3)(3a_{n-4} + 4a_{n-5}) + (3)(4)(3^2)(3a_{n-5} + 4a_{n-6}) \\
& + (3)(4^2)(3)(3a_{n-6} + 4a_{n-7}) + (4^3)(3a_{n-7} + 4a_{n-8}) \\
= {} & (3^3)(3)(a_{n-4}) + (3^3)(4)(a_{n-5}) \\
& + (3)(4)(3^2)(3)(a_{n-5}) + (3)(4)(3^2)(4)(a_{n-6})
\end{aligned}$$

$$
\begin{aligned}
&+ (3)(4^2)(3)(3)(a_{n-6}) + (3)(4^2)(3)(4)(a_{n-7}) \\
&+ (4^3)(3)(a_{n-7}) + (4^3)(4)(a_{n-8}) \\
= &(3^4)a_{n-4} + (3^3)(4)a_{n-5} \\
&+ (3)(3^3)(4)a_{n-5} + (3)(3^2)(4^2)a_{n-6} \\
&+ (3)(3^2)(4^2)a_{n-6} + (3)(4^3)(3)a_{n-7} \\
&+ (4^3)(3)a_{n-7} + (4^4)a_{n-8} \\
= &(3^4)a_{n-4} + (4)(3^3)(4)a_{n-5} \\
&+ (6)(3^2)(4^2)a_{n-6} + (4)(4^3)(3)a_{n-7} + (4^4)a_{n-8}.
\end{aligned}
$$

$$
\begin{aligned}
a_n =&(3)a_{n-1} + (4)a_{n-2} && \text{(Iteration 1)} \\
=&(3^2)a_{n-2} + (2)(4)(3)a_{n-3} + (4^2)a_{n-4} && \text{(Iteration 2)} \\
=&(3^3)a_{n-3} + (3)(4)(3^2)a_{n-4} + (3)(4^2)(3)a_{n-5} + (4^3)a_{n-6} && \text{(Iteration 3)} \\
=&(3^4)a_{n-4} + (4)(3^3)(4)a_{n-5} \\
&+ (6)(3^2)(4^2)a_{n-6} + (4)(4^3)(3)a_{n-7} + (4^4)a_{n-8}. && \text{(Iteration 4)}
\end{aligned}
$$

Notice anything special? The most obvious would be the coefficients of the first and last terms. The first term starts with a coefficient of $3^{(\text{iteration})}$. Then, for each term, that power of 3 fades into nothing as we move to the right. The same thing happens for 4, but going from right to left.

Then, you need to notice that the coefficients other than these ascending/descending coefficients of 3 and 4 make the binomial coefficients. Cool right? So that means these expansions follow Pascal's triangle and can be expressed as a sum of binomial coefficients as so:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (3^{n-k})(4^k)a_{-k}$$

Unfortunately, this is where we cannot go further. Notice that (1) we cannot simplify this into $(a+b)^n$ form because $a-k$ is not raised to a power of k and that (2) we would not be able to obtain any of these $a-k$ terms, since we are only given two initial conditions.

So with a second-order recurrence, the iteration method expands too many terms so even if we find a pattern, we are unable to generalize the a_x term to the initial conditions.

†

In the next section, we will discuss a similar method called the recursion tree method.

Exercises 5.5

Exercise 5.5.1

Use back-substitution to find a pattern and solve the following recurrence relation given $a_0 = 1$:

$$a_n = 3a_{n-1} + 2.$$

Exercise 5.5.2

Find the sequence $\{a_n\}$ that satisfies the recurrence relation using back-substitution:

$$a_n = 2a_{n-1} - 5, \quad a_1 = 4.$$

Exercise 5.5.3

Conjecture and solve the sequence $\{a_n\}$ which follows

$$a_n = 4a_{n-1} + n^2, \quad a_1 = 2.$$

Exercise 5.5.4

Given that $a_0 = 0$, use back-substitution to find a closed-form expression for the sequence

$$a_n = a_{n-1} + 3n.$$

Exercise 5.5.5

Solve

$$a_n = 5a_{n-1} + 7n + 1, \quad a_0 = 3$$

using back-substitution.

5.6 Recursion Tree Method

Let's do something different called the Recursion Tree Method.

Definition 5.6.1 (Recursion Tree Method)

The recursion tree method for linear recurrences depicts a visual representation that reveals the patterns of the recurrence in terms of divergence, which can be inspected through "rows" of branches.

Example 5.6.2

Say you are given the following code:

```
count = 0  # line 1
def T(n):   # line 2
    if(n == 0):  # line 3
        return 1  # line 4
    count += 1  # line 5
    count += T(n-1)  # line 6
    count += T(n-1)  # line 7
```

This code represents the non-homogeneous linear recurrence for the value of count after the function $T(n)$ is called. Find an explicit formula for the value of count after some call $T(n)$, for $n \geq 1$.

Solution.

$$T(n) = 2T(n-1) + 1, T(0) = 0,$$

where

$$\exists T(n) \forall n \geq 0 \text{ s.t. } n \in \mathbb{Z}.$$

Normally, this type of recurrence is used to solve for the runtime of these functions. For example, if we had a print statement instead of count, then we would be computing the total run time when you call any $T(n)$. But just note that you can treat $T(n)$ as our usual a_n.

To solve this relation, we can draw a recurrence tree graph. This graph will show each iteration and we will sum up how many times line 5 runs for, and how many $T(0)$'s end up in iteration n, when we finally reach $n = 0$. Consider the following graph in Figure 5.6.1 (assume that each branch represents a new recurrence, disregarding the same $T(n-3)$ that is being connected to as the same one):

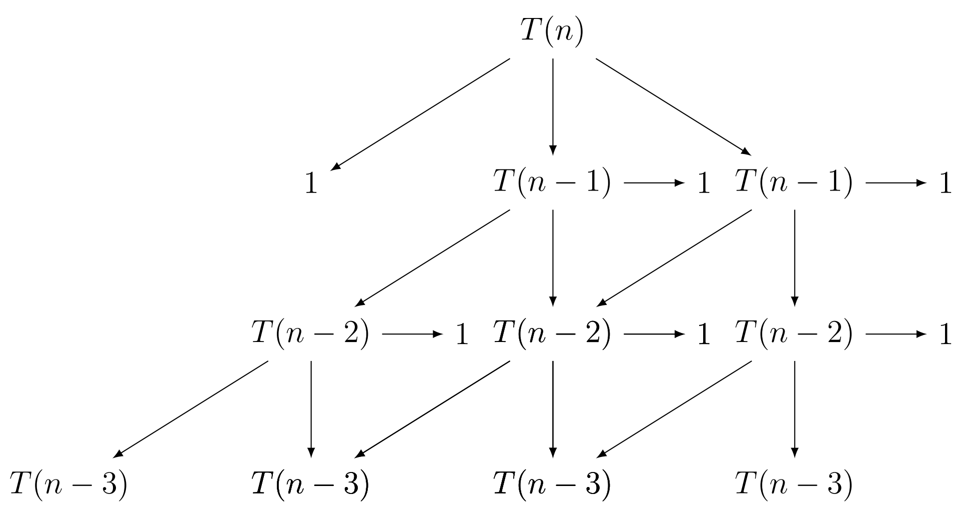


Figure 5.6.1: Recurrence Tree of $T(n)$

As you can see, the $T(n-k)$ appears from 2 times $\to$ 4 times $\to$ 8 times, and expands by a factor of 2 each time due to the $2T(n-1)$ in the recurrence, as shown in Table 5.6.1:

Iteration	1	2	3	...	n
Sum	1	2	4	...	2^{n-1}

Table 5.6.1: Iteration and Sum Progression of $T(n)$

Additionally, we can also tell that the 1 is appearing 1 time, 2 times, 4 times, 8 times, and so on. So that means that our final count when calling $T(n)$ will be the sum of 1's across all iterations and the sum of $T(0)$'s in the final iteration:

Sum of 1's:

$$1 + 2 + 4 + 8 + \cdots + 2^{n-1}.$$

Sum of T(0)'s in final iteration:

$$2^n.$$

Note that by this pattern, when $n = 1$, there are 21 terms of $T(n-1)$, when $n = 2$, there are 22 terms of $T(n-2)$—so when $n = n$, there should be 2^n terms of $T(n-n) = T(0)$):

$$T(n) = 1 + 2 + 4 + 8 + \cdots + 2^{n-1} + 2^n.$$

Using our geometric sum formula:

$$\sum_{i=0}^{n} a_1 r^i = \frac{a_1(1-r^n)}{1-r},$$

where $a_1 = 1, r = 2$, we acquire the desired explicit definition of $T(n)$:

$$\begin{aligned} T(n) &= \frac{1(1-2^n)}{1-2} \\ &= \frac{1-2^n}{-1} \\ &= \boxed{2^n - 1}. \end{aligned}$$

Therefore, we obtain that

$$T(n) = 2^n - 1.$$

†

Using the recursion tree method is just another representation that you may use to find a pattern, similar to the iteration method. This is a pretty simple method, and you can try to do this with homogeneous recurrences which will be even easier to do.

And that concludes all the methods I wanted to show you. There is also another method called the master method, feel free to explore that on your own. For the relevance of this topic, I will not go into it as it requires background knowledge from computer science algorithms/data structures class.

Recurrences are a huge deal within combinatorics, discrete math, and also computer science. While this book will not cover the deep theory behind more complicated recurrences, feel free to explore more on the Big O function dealing with non-linear or recurrences with log solutions.

Exercises 5.6

Exercise 5.6.1

Draw a recursion tree and find the solution to the following recurrence:

$$a_n = 2a_{n-1} + (-1)^n,$$

where $a_0 = 0$.

Exercise 5.6.2

Find the solution to the sequence a_n using the recursion tree method:

$$a_n = 3a_{n-1} + 7n^2, \quad a_1 = 2.$$

Exercise 5.6.3

Say $a_1 = 3$ for the linear non-homogeneous recurrence a_n. Use the recursion tree method to conjecture a pattern and formulate a closed-form expression for a_n, given that

$$a_n = 7a_{n-1} + n^3.$$

Exercise 5.6.4

A recurrence relation below can be solved in multiple ways. Solve by the recursion tree method, and then prove it using proof by induction:

$$a_n = 3a_{n-1} + 8n + 2, \quad a_2 = 4.$$

Exercise 5.6.5

Let

$$a_n = 2a_{n-1} + 5^n, \quad a_0 = 1.$$

Find the solution to this recurrence using the recursion tree method. Then, prove by induction.

5.7 Special Recurrence Examples

Now that we are equipped with many methods, try to solve some third-order linear homogeneous recurrence relations in the exercise at the end of this chapter. For those, I would suggest using either the characteristic root or generating functions method.

Now, let's explore how these recurrences can be applied to real-world examples.

> **Example 5.7.1**
> Find the recurrence relation for b_n, the total number of ways to create binary strings of n numbers with no consecutive 1s.

Solution. Binary numbers come in only 0s and 1s. As per recurrence relations, we want to look at the relationship every time you add another digit to the string:

$$b_0 \text{ (0 digits) } \rightarrow \text{ 1 way,}$$

there is one way to order nothing, by "the Principle of Nothing."

$$b_1 \text{ (1 digit) } \underline{\quad\quad} \rightarrow \text{ 2 ways,}$$

$$b_2 \text{ (2 digits) } \underline{\quad\quad} \quad \underline{\quad\quad} \rightarrow \text{ ? ways.}$$

Instead of brute counting the number of ways, let's see that if you add one more digit, there are two possible choices. One possible choice is for that digit would be 0. If it was 0, how many ways would there be to order everything? Well, it would be the exact same as b_1, since you already chose the number for the new digit, and there are no other restrictions. Now, if the new digit was 1, how many ways can you order it? Well, you cannot have consecutive 1s, so there is a restriction that the previous digit cannot be 1, and must be 0. Therefore, it pre-defines the last two digits if the new digit is 1, and therefore would be equal to the number of ways to order b_{n-2}, or in this case, b_0.

Here is a better visual:

Let this be $n-2$ digits:

____ ____ ____ ____ ____ ____ ____

Let this be $n-1$ digits:

____ ____ ____ ____ ____ ____ ____ ____

Let this be n digits:

____ ____ ____ ____ ____ ____ ____ ____ ____

The case for n digits can be split into two cases, where the nth digit is 0 or 1:

____ ____ ____ ____ ____ ____ ____ ____ __0__

____ ____ ____ ____ ____ ____ ____ ____ __1__

If we count the number of ways for each of these cases and add them together, we will obtain the total number of ways with no restrictions, since we know that the last digit must be either 0 or 1, and we just split it.

So, when the nth digit is 0, we have:

____ ____ ____ ____ ____ ____ ____ ____ __0__ ,

where the rest $n-1$ digits do not have any restrictions, since 0 does not potentially violate any of the restrictions, which was having consecutive 1s. So, we have b_{n-1} ways of obtaining this.

When the nth digit is 1, we have:

____ ____ ____ ____ ____ ____ ____ ____ __1__ .

This time, we have a potential violation for if the second to last digit is equal to 1. Since we want to get the total number of ways to do this, we need to make sure all cases we count are not violating any rules. Therefore, the second to last digit must be 0 for it to be possible. This is the same as counting the number of ways to create the binary string with these two spaces already filled in:

____ ____ ____ ____ ____ ____ ____ __0__ __1__,

which leaves $n-2$ spaces with no restrictions since 0 again does not violate any rules. Therefore, the number of ways to obtain this binary string when the nth digit is 1 would be equal to b_{n-2}.

Adding both cases together, we get

$$b_n = b_{n-1} + b_{n-2}.$$

†

Notice this is the Fibonacci sequence. We already solved this with generating functions, but I encourage you to solve this recurrence using other methods like the characteristic root method, or try to prove Binet's formula correct by using proof by induction.

Example 5.7.2

(Try yourself) How many ways are there to tile a $1xn$ board with only 1×1 dominoes and/or 1×2 dominoes? Hint: check how we solved the previous question.

Example 5.7.3

Let a_n equal the number of ways to park cars in a row with n spaces using Boyota, Ronda, and Saudi. A Boyota or Ronda requires two spaces, whereas a Saudi requires only one space.

1. Find an for $n = 1, 2, 3, 4, 5$.
2. Find and solve a recurrence relation for this problem.

Solution. Well, this is quite similar to our last problems, right? We use the trick where we look at the last car parked.
(a) Let’s start by looking at $n = 1$.
If $n = 1$, then only Saudi can park:

$$a_1 = 1.$$

If $n = 2$, you can either park with Boyota, Ronda, or both spaces with Saudi. Thus, for $n = 2$, the number of ways to park cars in 2 spaces is given by

$$a_2 = 3.$$

If $n = 3$, let’s denote Boyota, Ronda, and Saudi by B, R, and S, respectively. So if $n = 3$ spaces, we can have SSS, BS, RS, SR, or SB. That is 5 ways:

$$a_3 = 5.$$

If $n = 4$, then we can have SSSS, SSB, SSR, RSS, BSS, SRS, SBS, RR, RB, BR, or BB, or 11 ways:

$$a_4 = 11.$$

If $n = 5$, we can have SSSSS, SSSB, SSSR, RSSS, BSSS, SRSS, SBSS, SSRS, SSBS, BBS, BSB, SBB, RRS, RSR, SRR, RBS, RSB, SRB, BRS, BSR, or SBR. That is 21 ways:

$$a_5 = 21.$$

(b) We now must find the recurrence relation and then solve for the explicit formula. Say we have n spaces:

____|____|____|____|____|____|____|____→ ”$n - 2$” spaces

____|____|____|____|____|____|____|____|____→ ”$n - 1$” spaces

____|____|____|____|____|____|____|____|____|____→ ”n” spaces

Let’s look at the last car parked in the “n” spaces. The three possible ways to park those two spaces are to have a Boyota, a Ronda, or one Saudi:

____|____|____|____|____|____|____|____|____Boyota____→ ”n” spaces

____|____|____|____|____|____|____|____|____Ronda____→ ”n” spaces

____|____|____|____|____|____|____|____|____|__Saudi__→ ”n” spaces

For the two instances that Boyota or Ronda are parked last, there are $n - 2$ spaces left, since they take up two spaces. These two spaces have

$$a_{n-2}$$

ways of being parked. For the instance that Saudi is parked last, $n - 1$ spaces are remaining, leaving

$$a_{n-1}$$

ways to be parked. Since these three instances all add up to the total number of ways to park these cars in n spaces, we have:

$$a_n = a_{n-1} + 2a_{n-2}.$$

Remember also that from part (a),

$$a_1 = 1, a_2 = 3.$$

Setting everything to the left side, we get:

$$a_n - a_{n-1} - 2a_{n-2} = 0.$$

Now, let's use the characteristic root method. We use the lambda hypothesis, $a_n = \lambda^n$:

$$\lambda^n - \lambda^{n-1} - 2\lambda^{n-2} = 0,$$

$$\lambda^{n-2}(\lambda^2 - \lambda - 2) = 0.$$

The characteristic polynomial is $\lambda^2 - \lambda - 2$, which we can factor:

$$\lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Our characteristic roots are then

$$\lambda = 2, -1,$$

and so

$$\lambda^n = 2^n, (-1)^n = a_n.$$

Since the roots are distinct, we do not have to worry about the root functions being linearly independent. Therefore, we obtain the general solution form:

$$a_n = A(2^n) + B(-1)^n.$$

Using our given, $a_1 = 1, a_2 = 3$, we can construct our final solution by finding A, B.

$n = 1$:

$$1 = A(2^1) + B(-1)^1,$$

$$1 = 2A - B.$$

$n = 2$:

$$3 = A(2^2) + B(-1)^2,$$

$$3 = 4A + B.$$

Adding the two equations together to cancel out B, we derive the value for A:

$$4 = 6A, A = 2/3.$$

Let us plug in A to the first equation we got when we set $n = 1$ to find B:

$$1 = 2(2/3) - B,$$

$$\begin{aligned} B &= 4/3 - 1 \\ &= 1/3. \end{aligned}$$

Therefore,

$$\begin{aligned} a_n &= (2/3)(2^n) + (1/3)(-1)^n \\ &= \frac{(2)2^n + (-1)^n}{3} \\ &= \boxed{\frac{2^{n+1} + (-1)^n}{3}}. \end{aligned}$$

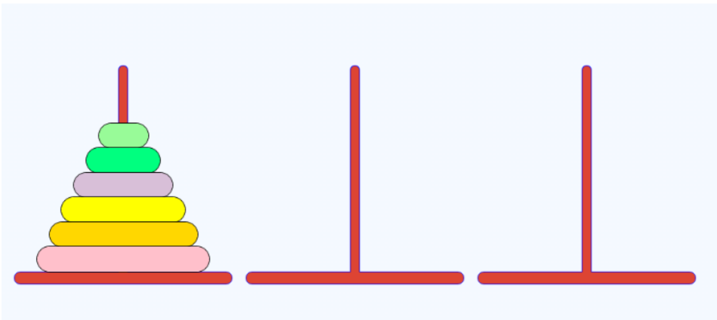
Let's test this just before we move on to our last example. We know that $a_3 = 5$ from our testing from part (a):

$$a_3 = 5,$$

$$\begin{aligned} a_3 &= \frac{2^{n+1} + (-1)^n}{3} \\ &= \frac{2^{3+1} + (-1)^3}{3} && \text{(Plugging in } n = 3) \\ &= \frac{2^4 - 1}{3} \\ &= \frac{16 - 1}{3} \\ &= \frac{15}{3} \\ &= \boxed{5}. \checkmark \end{aligned}$$

†

Great, onto our last example, which is a fun example as a game puzzle. This one is a very common real-world problem that can be solved through recurrences. Specifically, it is a game that you can play. I suggest you try this out either now or sometime soon and see if you reach the best score. Click this link to try it out: https://www.mathsisfun.com/games/towerofhanoi.html (This will be the website that is used to obtain images in this section.)



The game is called the Tower of Hanoi. In this puzzle, there are three poles as shown above, and you must move some n disks to the center pole, in the same order it is in right now.

Example 5.7.4

Find and solve a recurrence relation

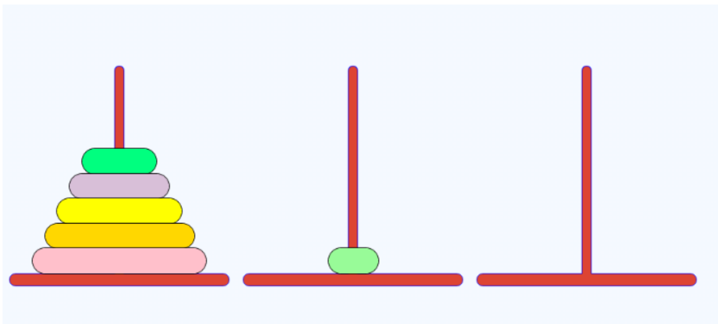
$$h_n$$

of the minimum number of moves it would take to move n disks in the Tower of Hanoi (starting from either the left or right pole) in a tower order (i.e. widest on the bottom and descending to the least widest on the top) to the center pole stacked in also the tower order. You may never stack a disk onto a pole where the top disk is less wide than it.

Solution. This is quite a daunting problem to look at initially, which is why I encourage you to play the game first to get a feeling of what the puzzle is all about. The trick here is to look at each recurrence step, or in this case, each case of n disks.

For $n = 1$, we know that we can achieve this within one move, by simply placing the disk onto the middle pole:

(I will use six disks temporarily for $n = 1$ and $n = 2$ since it was not available on the website. But even in this case, it contains one disk on top so you can imagine the other five disks do not exist, as we will never move them)

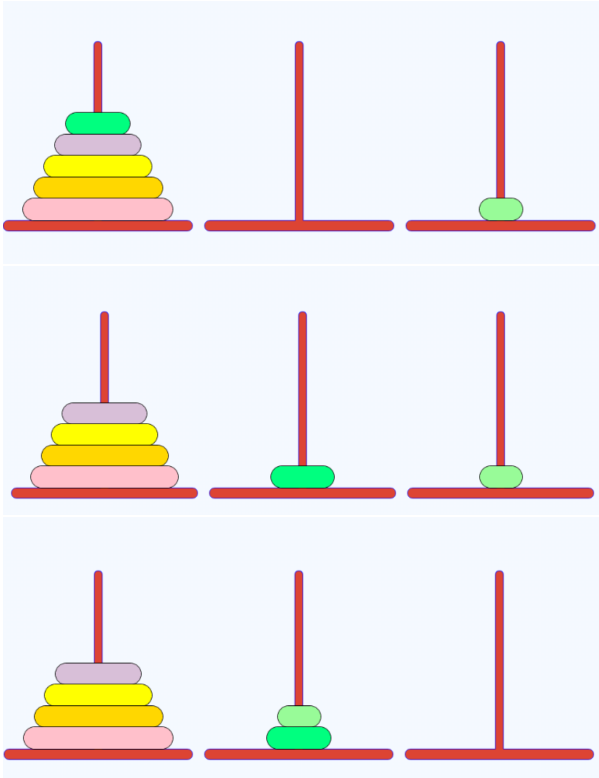


Thus, $h_1 = 1$.

For $n = 2$, instead of placing the d_1 (denote $d_1, d_2, \ldots, d_n$ as each disk, with d_1 being the smallest disk, d_2 being the second smallest, and so on) immediately to the center pole, we can first move d_1 to the right pole and then move the second disk to the center pole, and then move d_1 back to the center pole. Let's denote the left pole as p_1, the center pole as p_2, and the right pole as p_3. So we define our sequence of moves to be the minimum amount for $n = 2$:

$$d_1 \to p_3, d_2 \to p_2, d_1 \to p_2.$$

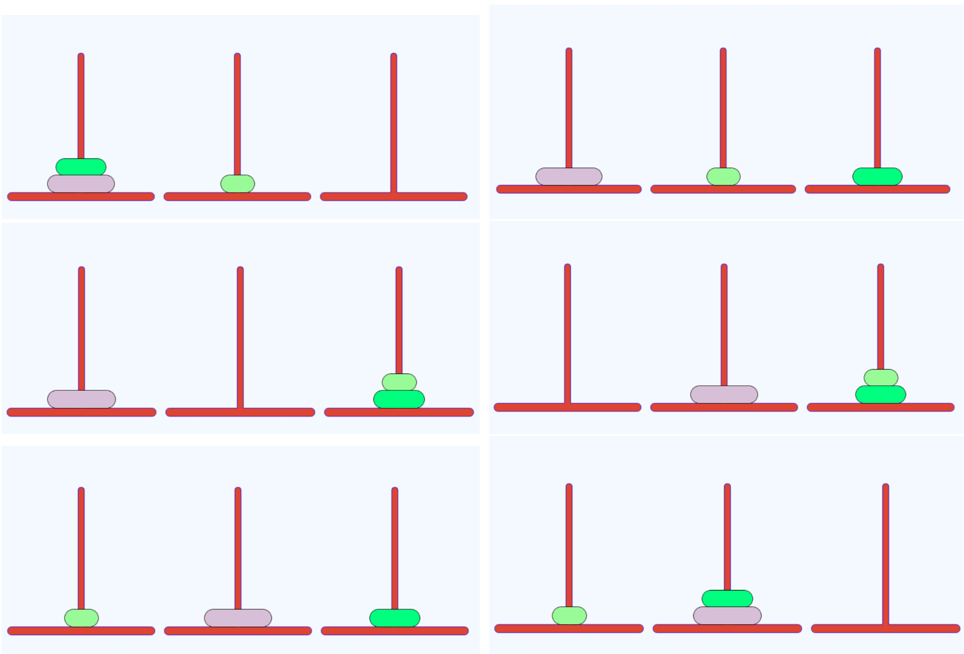
This move-set is shown below:

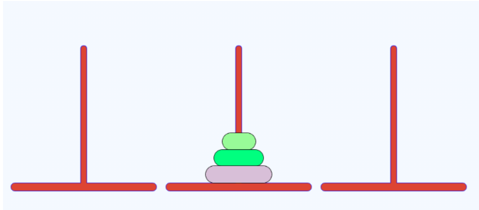


This is three minimum ways to achieve this two-stack, and so $h_2 = 3$.
For $n = 3$, we would have to do

$$d_1 \to p_2, d_2 \to p_3, d_1 \to p_3, d_3 \to p_2, d_1 \to p_1, d_2 \to p_2, d_1 \to p_2.$$

This is demonstrated below (left to right, then down to start at the left of the row below):





We find that $h_3 = 7$.

Here is where we need to realize a pattern. The trick here is that you see we have obtained the stack for $n = 2$ in the third image (top right), but it is just on pole 3 instead of pole 2. Then, look at those two same disks again, after placing d_3 down to pole 2, we do an operation to get the same two disks d_1 and d_2 to the center pole. Notice that this operation is the same as moving those from its initial stack on pole 1 to pole 3.

This is the most important thing to realize about this problem. For every single puzzle we do for some n disks, we can think of it as trying to get n-1 towered disks (which contain $d_1, d_2, \ldots, d_{n-1}$) to pole 3, placing disk d_n to pole 2, and then getting the same $n-1$ towered disks (which contain $d_1, d_2, \ldots, d_{n-1}$) from pole 3 to pole 2, completing the puzzle.

This means that we do the operation needed for moving $n-1$ towered disks to another pole twice.

So: First, we move $n-1$ towered disks to pole 3. This is denoted by h_{n-1}. Second, we move disk d_n to pole 2. This is denoted by 1, as it is one extra move we take. Third, we move all the towered disks to pole 2. This is the same operation as (1), therefore it is also denoted by h_{n-1}.

$$\therefore h_n = 2h_{n-1} + 1.$$

And here are our initial conditions that we found,

$$h_1 = 1, h_2 = 3, \text{and} h_3 = 7.$$

(Note that using $h_0 = 1$ by the principle of nothing is technically incorrect. This is because the Principle of Nothing says that moving nothing means that no moves are required, and counts as an arrangement. This question is not asking for the number of ways for it to be arranged, but it is asking for the number of moves required to place zero disks to pole 2. Therefore, the Principle of Nothing would instead say that there are no moves required to place zero disks to pole 2.)

Anyway, this is now sufficient enough to solve the recurrence.

Let us use the iteration method.

$$h_n = 2h_{n-1} + 1,$$

$$h_{n-1} = 2h_{n-2} + 1.$$

Substituting in for h_{n-1} in the main equation:

$$h_n = 2(2h_{n-2} + 1) + 1,$$

$$= (2)(2)h_{n-2} + 2 + 1,$$
$$= 2^2 h_{n-2} + 2 + 1. \qquad \text{(Let this be eq. 1)}$$

Next, the original recurrence also gives h_{n-2}:

$$h_{n-2} = 2h_{n-3} + 1.$$

Substitute in for h_{n-2} in eq. 1:

$$\begin{aligned} h_n &= 2^2(2h_{n-3} + 1) + 2 + 1, \\ &= (2^2)(2)h_{n-3} + 2^2 + 2 + 1, \\ &= 2^3 h_{n-3} + 2^2 + 2 + 1. \qquad \text{(Let this be eq. 2)} \end{aligned}$$

Another one, and we find that

$$h_{n-3} = 2h_{n-4} + 1.$$

Substituting for h_{n-3} into eq. 2:

$$\begin{aligned} h_n &= (2^3)(2h_{n-4} + 1) + 2^2 + 2 + 1, \\ &= (2^3)(2)h_{n-4} + 2^3 + 2^2 + 2 + 1, \\ &= 2^4 h_{n-4} + 2^3 + 2^2 + 2 + 1. \qquad \text{(eq. 3)} \end{aligned}$$

I think we can see a pattern here. Let's rewrite eq. 1, eq. 2, eq. 3 in an equal form using the fact that $1 = 2^0$ and $2 = 2^1$:

$$\begin{aligned} h_n &= 2^2 h_{n-2} + 2^1 + 2^0, && \text{(eq.1)} \\ h_n &= 2^3 h_{n-3} + 2^2 + 2^1 + 2^0, && \text{(eq.2)} \\ h_n &= 2^4 h_{n-4} + 2^3 + 2^2 + 2^1 + 2^0. && \text{(eq.3)} \end{aligned}$$

Thus, we conjecture the pattern:

$$h_n = (2^k)h_{n-k} + 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2^2 + 2^1 + 2^0.$$

We use the sum of geometric series formula to simplify the sum on the right, where the geometric sum formula is as follows:

$$\sum_{i=0}^{n} a_1 r^i = \frac{a_1(1 - r^n)}{1 - r},$$

where a_1 is the first term of the geometric sum, r is the common ratio, and n is the number of terms.

Note that based on the communitive property of addition,

$$2^{k-1} + 2^{k-2} + 2^{k-3} + ... + 2^2 + 2^1 + 2^0$$

$$= 2^0 + 2^1 + 2^2 + ... + 2^{k-3} + 2^{k-2} + 2^{k-1}.$$

Using $a_1 = 1, r = 2$, and $n = k$, we get:

$$\begin{aligned} 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2^2 + 2^1 + 2^0 &= \frac{1(1-2^k)}{1-2} \\ &= \frac{1-2^k}{-1} \\ &= 2^k - 1. \end{aligned}$$

Therefore,

$$h_n = (2^k)h_{n-k} + 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2^2 + 2^1 + 2^0$$

simplifies to

$$h_n = (2^k)h_{n-k} + 2^k - 1.$$

Now, we know that $h_1 = 1$, so we will try to get $h_1 = h_{n-k}$ in order to substitute in and get a formula in terms of only n. To do this, $n - k$ must equal to 1:

$$n - k = 1,$$

$$n = k + 1,$$

$$k = n - 1.$$

Finally, substitute for k for $n - 1$ to obtain an explicit formula:

$$\begin{aligned} h_n &= (2^{n-1})h_{n-(n-1)} + 2^{n-1} - 1 \\ &= (2^{n-1})h_{n-n+1} + 2^{n-1} - 1 \\ &= (2^{n-1})h_1 + 2^{n-1} - 1 \\ &= (2^{n-1})(1) + 2^{n-1} - 1 \\ &= 2^{n-1} + 2^{n-1} - 1 \\ &= (2^1)2^{n-1} - 1 \\ &= \boxed{2^n - 1}. \end{aligned}$$

Therefore,

$$h_n = 2^n - 1.$$

(Feel free to prove with proof by induction.) †

I encourage you to try it with $n - k = 2$ instead so that you can plug in $h_2 = 3$, or even $h_3 = 7$. It will also work out to the same answer.

Let's test our formula—which we already know $h_2 = 3, h_3 = 7$ from explicitly listing all the cases for the minimum number of ways. By our formula,

$$\begin{aligned} h_2 &= 2^2 - 1 \\ &= 4 - 1 \\ &= \boxed{3}, \ \checkmark \end{aligned}$$

$$\begin{aligned} h_3 &= 2^3 - 1 \\ &= 8 - 1 \\ &= \boxed{7}. \ \checkmark \end{aligned}$$

Great, this is a very fun puzzle and application of recurrence relations. Next time, challenge your friend to this game and you can now confidently bet them that with however many disks they have, they cannot complete the puzzle any less than $2^n - 1$ moves.

Recurrences have a wide range of applications, spanning from binary strings/tiling problems (Fibonacci Sequence), parking problems, and now the Tower of Hanoi puzzle. By mastering methods such as characteristic roots, generating functions, proof by induction, iteration, and recursion trees, you have built a very useful toolkit for tackling these problems—and even beyond theoretical mathematics into practical applications in computer science, engineering, and beyond.

To further solidify your understanding, you can explore the exercise problems on the next page. These problems will test the skills that you have learned in this chapter and they will challenge you with additional steps, encouraging you to apply and expand upon your knowledge. Solutions are provided afterward to help you verify your solutions and provide help if needed.

Happy problem-solving!

Exercises 5.7

Exercise 5.7.1

The Gambler's Ruin Problem: Two gamblers (Gambler A and Gambler B) play a game repeatedly. During each round, gambler A has a probability p to win a dollar while $q = 1 - p$ chance to lose 1 dollar (note that this is equal to the chance for Gambler B to win and likewise, p is the probability that Gambler B loses). Assume that Gambler A initially has i dollars and B has some $N - i$ dollars (for $N, i \notin \mathbb{Z}^-$).

If the game ends when one of the gamblers has no more money to gamble (with the other gambler having N dollars), find the probability p_i that Gambler A wins the game (first find a recurrence relation equation, then an explicit formula containing N and i). Use the result to conjecture the probability of profits for a gambler who plays repeatedly at a casino, which usually contains N much greater than i.

Exercise 5.7.2

Let a_n be the number of sequences of 1's, 3's, and 5's whose terms sum to n.

1. Find a_n for the values $n = 1, 2, 3, \dots, 7$.
2. Find and solve a recurrence relation for a_n.

Exercise 5.7.3

Recall that one gumball machine near your local grocery store. Suppose that the very first time that a quarter is put into the machine, only 1 gum comes out. The second time pops out 4 gumballs, while the third time is 16 gumballs, the fourth time 64 gumballs, etc.

Find both a recursive and a closed, explicit formula using the Characteristic Root technique for the number of gumballs that the nth customer will get (assuming everyone buys once).

Exercise 5.7.4

Let a_n be the number of $1 \times n$ tile patterns that can be made using 1×1 squares, which come in 4 different colors, and 1×2 dominoes, which have 5 distinct colors. Find a recurrence relation to model this problem. Then, find the first 6 terms of this sequence $(a_n)_{n \geq 1} = a_1, a_2, \dots$. Finally, solve this recurrence relation using any technique to find a closed formula for a_n.

Exercise 5.7.5

You can use 1×1 tiles that can either be blue or red or 1×2 tiles that can be either green, orange, or purple. Find a recurrence relation and then solve it for the number of distinct $1 \times n$ path designs that can be made using these tiles.

CHAPTER 6

Partition Theory

6.1 Partitions of Integers

Example 6.1.1

Take any positive integer $n \geq 1$ and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n,$$

where $\lambda_i \geq 1$, then how many *unordered* solutions are there?

Recall that this is similar to stars and bars formula from Theorem 1.4.3, where we had the solution to be $\binom{k-1}{n-1}$. However, this is not the same problem, as the stars and bars theorem is made for *ordered* solutions and we want unordered.

A better way to represent this problem, then, would to be use *weakly decreasing* tuples of integers as the λ_i's.

Definition 6.1.2 (Weakly Decreasing Integers)

A sequence of weakly decreasing integers can be defined by a sequence generated in which each pair of consecutive terms always has a non-negative difference. Functions that generate these sequences are considered monotonic functions.

We have the following representation, noting that ordering these lambdas as weakly decreasing naturally restricts the sequences to be unordered:

$$\{\{\lambda_1, \lambda_2, \ldots, \lambda_k\}\} = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$$

This is called a partition of an integer.

Definition 6.1.3 (Integer Partitions)
A partition of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that (1) they are weakly decreasing and (2) the sum of λ_i adds up to n.

That is,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k,$$
$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

Note that

1. each λ_i is called a "part" of n,
2. the number of parts in a particular partition λ, or the length of λ, can be denoted by $\ell(\lambda)$, and
3. the partition λ of n is given by $\lambda \vdash n$, where $|\lambda|= n$.

An example of a partition of the integer 9 would be $(3, 3, 2, 1)$. Notice that this is both weakly decreasing and all the λ_i add up to 9.

Example 6.1.4
Find all the partitions of the integer $n = 5$.

Solution. We can start with the first part being equal to 5. We have one possibility here:

$$(5).$$

Next, we try $\lambda_1 = 4$. We again have one option:

$$(4, 1).$$

$\lambda_1 = 3$ allows for two options:

$$(3, 2),$$
$$(3, 1, 1).$$

$\lambda_1 = 2$ yields another 2 partitions:

$$(2, 2, 1),$$

$$(2, 1, 1, 1).$$

$\lambda_1 = 1$ has one possible partition:

$$(1, 1, 1, 1, 1).$$

There are 7 such partitions. †

Now circling back to the original question at the start of this section is the count of partitions possible for a certain integer n. We can represent this as $P(n)$, counting the number of partitions for a number n. In the previous example, we found that $P(5) = 7$. Let's list out some more partitions of other integers to see if there is a pattern and perhaps we can extract a closed formula for this.

Partitions of 1:

1. (1)

Partitions of 2:

1. (2)
2. (1,1)

Partitions of 3:

1. (3)
2. (2,1)
3. (1,1,1)

Partitions of 4:

1. (4)
2. (3,1)
3. (2,2)
4. (2,1,1)
5. (1,1,1,1)

A partition of 0 is also possible, and the $P(0)$ is actually equal to 1. The Principle of Nothingness says that there is only one way to do nothing, therefore $P(0) = 1$.

So we have

$$P(0) = 1, P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5, P(5) = 7.$$

This looks quite similar to the Fibonacci Sequence but it is not quite it.

Recall that compositions of integers are ordered sequences, where each of its parts sum up to the given integer. In the previous chapter, we concluded that the integer 4 can be written like the following 8 compositions:

1. $1 + 1 + 1 + 1$
2. $1 + 1 + 2$
3. $1 + 2 + 1$
4. $2 + 1 + 1$
5. $3 + 1$
6. $1 + 3$
7. $2 + 2$
8. 4

On the other hand, we previously found that the partitions of 4 include the following:

1. 1+1+1+1
2. 2+1+1
3. 3+1
4. 2+2
5. 4

Notice that since partitions are unordered sequences, one partition will often model several orderings that are equal by rearranging. In this case, the second item from the partitions of 4 covers items 2-4 from the compositions and item 3 from partitions covers items 5-6 for compositions.

Compositions

$$\underbrace{1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1}_{\text{2+1+1 (partition)}} \quad | \quad \underbrace{3 + 1, 1 + 3}_{\text{3+1 (partition)}}$$

$$\underbrace{1 + 1 + 1 + 1}_{\text{1+1+1+1 (partition)}} \quad | \quad \underbrace{2 + 2}_{\text{2+2 (partition)}} \quad | \quad \underbrace{4}_{\text{4 (partition)}}$$

For the integer 4, there are 8 compositions (ordered sums) and 5 partitions (unordered sums).

Definition 6.1.5 (Partitioning Integer into n Parts)
The number of ways to partition a positive integer k into n parts is denoted by $P(k,n) = P_k(n)$. By definition,

$$P(k,n) = P_k(n) = |\{\lambda \vdash k | \ell(\lambda) = n\}|.$$

Note that this may sometimes be expressed as how permutations are expressed as $(P(n,k))$, which is why some prefer $P(k,n)$.

For instance, if we wanted to find $P(7,3)$, then that would be:

$$\begin{aligned}
5+1+1 &\to (5,1,1),\\
4+2+1 &\to (4,2,1),\\
3+3+1 &\to (3,3,1),\\
3+2+2 &\to (3,2,2).
\end{aligned}$$

This means that $P(7,3) = \boxed{4}$, as there are four partitions of 7 that have length 3.

Unfortunately, there are no pretty closed formulas for calculating either $P(n)$ or $P(k,n)$, but we can think of $P(k,n)$ combinatorially as separating some integer k into a composition or a sum of 1's. For instance, $n = 6$ can be expressed as $1+1+1+1+1+1$. Now, if we wanted to find partitions with three parts, which means $k = 3$, then we can write the following equation:

$$1+1+1+1+1+1 = x_1 + x_2 + x_3,$$

where each of the plus signs on the right side behaves like a bar ($|$). In that case, we have

$$1+1+1+1+1+1 = x_1|x_2|x_3.$$

We are now essentially trying to take six identical balls and placing them in three identical bins (to account for unordered sequences), where each bin is not empty ($\lambda_i \geq 1$).

From this, the partition of an integer k would be equal to the sum of all possible partitions of k into n non-empty boxes, or parts. In other words,

$$P(k) = \sum_{i=1}^{n} P(k,i).$$

Lemma 6.1.6 (Partition Decomposition)
Let $P(k)$ be the number of partitions of a positive integer k and $P(k,n)$ be the number of partitions of k into exactly n parts. Then the total number of partitions of k is given by the sum of the number of partitions into all possible n parts:

$$P(k) = \sum_{i=1}^{n} P(k,i)$$

Before we move on, let's take a look a specific case of integer partitions.

Definition 6.1.7 (Partitions into Distinct Parts)

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with distinct parts (alternatively, $\lambda_1 > \lambda_2 > \cdots > \lambda_k$) can be represented by $Q(k, n)$.

$$Q(k, n) = \# \text{ of partitions of an integer } k \text{ into } n \text{ distinct parts.}$$

For example, we previously found that

$$(5, 1, 1), (4, 2, 1), (3, 3, 1), (3, 2, 2)$$

were all the partitions of 7 into 3 parts. We see that only one of them have all distinct parts, which is $(4, 2, 1)$. Therefore, $Q(7, 3) = |\{(4, 2, 1)\}| = \ell(\{(4, 2, 1)\}) = \boxed{1}$.

Exercises 6.1

Exercise 6.1.1

Find the number of partitions of 8, $P(8)$.

Exercise 6.1.2

Find the number of partitions of 9 into 5 parts, $P(9, 5)$.

Exercise 6.1.3

Find the number of partitions of $n = 9$ where each part is an odd number.

Exercise 6.1.4

Compute $Q(9, 6)$.

Exercise 6.1.5

Find the number of partitions of the integer $n = 12$ into exactly 4 parts where each part is greater than or equal to 3.

Exercise 6.1.6

Compute $Q(15, 5)$.

Exercise 6.1.7

Determine the number of partitions of $n = 10$ such that one of the parts must be exactly 3.

Exercise 6.1.8

Compute the number of partitions of $n = 18$ into parts where each part is a prime number.

Exercise 6.1.9

Determine the number of partitions of $n = 6$ into exactly 3 parts where the first part is not greater than 2.

Exercise 6.1.10

Find the number of partitions of $n = 11$ into 4 distinct parts where the smallest part is 2.

Exercise 6.1.11

Suppose $f(n, k)$ is the number of ways to distribute k chocolate bars to n children such that each child is given at most two bars. Here are some examples: $f(3, 7) = 0$, $f(3, 6) = 1$, and $f(3, 4) = 6$. Find the value of $f(2006, 1) + f(2006, 4) + f(2006, 7) + \cdots + f(2006, 4012)$.

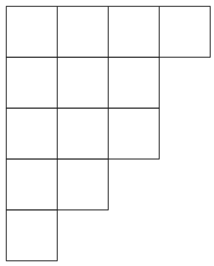
Exercise 6.1.12

For any partition π, let $A(\pi)$ denote the number of 1's that appear in π, and denote $B(\pi)$ as the number of distinct integers that show up in π. For example, if $n = 13$ and π represents the integer partition $1 + 1 + 2 + 2 + 5$, then $A(\pi) = 2$, $B(\pi) = 3$. Show that for a fixed number n, the sum of $A(\pi)$ over all possible partitions π of n is the same as the sum of $B(\pi)$ over all possible partitions π of n.

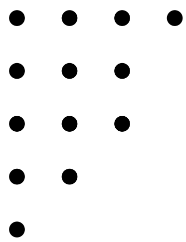
6.2 Ferrers Diagram

The last chapter discussed the term of weakly decreasing ordered integers that are, together, a partition of a certain integer. What if we represented each of these parts instead with a unit square block, where each part corresponds to its own row and the weakly decreasing order can be seen going from top to bottom?

For example, if we had the specific partition of 13 $(4, 3, 3, 2, 1)$, then the first part, 4, would become four unit squares in one row. Then, below that row, the second part would be represented with three unit square blocks lined horizontally adjacent (i.e. in the same row). This is performed for every part, creating the following figure:



As you can see, this figure has the area of the integer it is a partition of, which is 13. This can also be represented with dots instead of unit squares, but for this section, we will be primarily using the unit square representation.



This is known as the Ferrers diagram. Note that a bijection exists between partitions of a positive integer n and Ferrers diagrams with n blocks.

Definition 6.2.1 (Ferrers Diagram)

A Ferrers diagram, also known as Young diagram or Young tableau, is a graphical representation of a partition of a positive integer n. It consists of rows of boxes, where each row represents a part of the partition.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ (weakly decreasing) and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$, the Ferrers diagram can be constructed by the rules below:

- The diagram has k rows.
- The i-th row contains λ_i boxes, with all rows aligned to the left.
- The diagram is arranged such that each row does not exceed the length of the row above it.

An alternative definition is shown below using the notation $\mathrm{FD}(\mu)$.

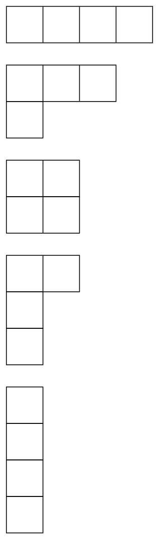
$$\mathrm{FD}(\mu) = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} \;\middle|\; \begin{matrix} 1 \le i \le k \\ 1 \le j \le \mu_i \end{matrix} \right\}$$

Example 6.2.2

List all the partitions (represented by μ) of 4 using Ferrers diagram.

Solution. There are 5 partitions of 4: $\mu = (4), (3,1), (2,2), (2,1,1)$, and $(1,1,1,1)$.

We can now draw the Ferrers diagrams for each of these partitions:

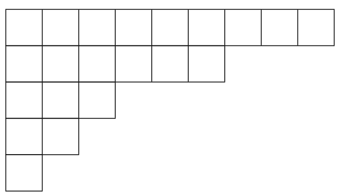


†

Again, since each box has a unit area (1),

$$\mathrm{FD}(\mu) = \mathrm{Area}(\mu) = |\mu| = n.$$

There is an interesting feature of Ferrers diagram. Inspect the following Ferrers diagram for a partition of 21:



Do you see anything? Let's draw a diagonal line $y = -x$ where $(0,0)$ is at the upper left corner block, as shown in Figure 6.2.1 below.

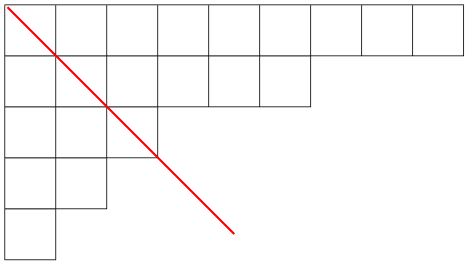


Figure 6.2.1: Ferrers Diagram Diagonal

What happens if we reflect this figure over that line? Aha! We achieve what's called a self-conjugate, which sets the first part of the Ferrers diagram partition as the length of the partition, or $\ell(\mu)$. This is shown in Figure 6.2.2.

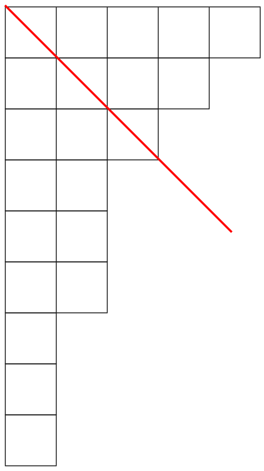


Figure 6.2.2: Reflected Ferrers Diagram

This new Ferrers diagram satisfies the partition requirements; it is weakly decreasing and has the same amount of area as the partition number, $n = 21$, because reflections are rigid transformations. Therefore, we have that $(5, 4, 3, 2, 2, 2, 1, 1, 1)$ is a partition of 21 (and indeed, $5 + 4 + 3 + 2 + 2 + 2 + 1 + 1 + 1 = 21$).

Every partition can be reflected to form either the same or a different partition. These are called conjugate partitions.

Definition 6.2.3 (Conjugate Partitions)

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a partition of a positive integer n, where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_k > 0$. The conjugate of μ, denoted μ', is the partition where μ'_j is equal to the number of parts in μ that are greater than or equal to j. Then μ is called a conjugate partition if $\mu = \mu'$. In other words,

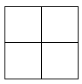
$$\mu_i = |\{j : \mu_j \geq i\}| \quad \text{for all } 1 \leq i \leq k,$$

$$\ell(\mu) = \mu'_1, \quad \text{and} \quad \ell(\mu') = \mu_1.$$

Note that $P^*(k, n) = P(k, n)$, where $P^*(k, n)$ denotes the number of conjugate partitions of k with exactly n parts, and $P(k, n)$ denotes the number of partitions of k into exactly n parts. This is true because each partition has a unique conjugate partition, and a bijection is established between the set of all partitions and the set containing all conjugate partitions. The number of partitions into k parts is invariant under the conjugation (or reflection) operation.

Furthermore, the conjugate of a partition is considered an involution, meaning that the conjugate of the conjugate of a partition μ is μ itself. That is, $(\mu')' = \mu$. This property further ensures that conjugation is a bijection from the set of partitions to itself.

In many cases, a special kind of conjugate partition is evident, such as partitions of 4 into two parts (2,2), which creates the Ferrers diagram below.



Such partitions are called self-conjugate partitions because they are symmetric reflecting over the diagonal line.

Definition 6.2.4 (Self-Conjugate Partitions)

A partition μ is called a self-conjugate partition if it is equal to its conjugate (denoted by SC(n). That is,

$$\text{SC}(n) = \{\mu \vdash n | \mu = \mu'\}.$$

Self-conjugate partitions are fixed points of the conjugate involutions. This means that if you pair up each partition with their conjugate partitions, then self-conjugate partitions will be left.

In general, proving these numbers for n with a closed formula is hard, which we will explore in the next section.

Let's take a look at two more things before we move on to the next section.

Example 6.2.5

Consider a box with m columns and n rows. We want to count the number of Ferrers diagrams (partitions) that can fit inside this box. Each partition corresponds to an integer k, where $0 \leq k \leq m \times n$, and satisfies the following conditions:

1. The partition has at most n parts.
2. Each part of the partition is of size at most m.

How many such partitions exist?

Solution. A $m \times n$ lattice graph can be constructed. All Ferrers diagrams can be fixed starting on the leftmost corner. This can be thought of as paths from the bottom left corner (point A) to the top right corner (point B), in which fixing all Ferrers diagrams at the top left corner ensures there are no repeated paths. Therefore, since there is a bijection between possible partitions (or Ferrer's diagrams), the answer is equal to the number of paths that can be formed from point A to point B.

Formed paths inform a Ferrers diagram by separating the graph into two parts, with the Ferrers diagram being the upper left part. These paths must only go right or upwards, since allowing it to go left or downwards would break the weakly decreasing requirement of Ferrers diagram and partitions. One such path is shown below in Figure 6.2.3, where a Ferrers diagram in cyan can also be seen.

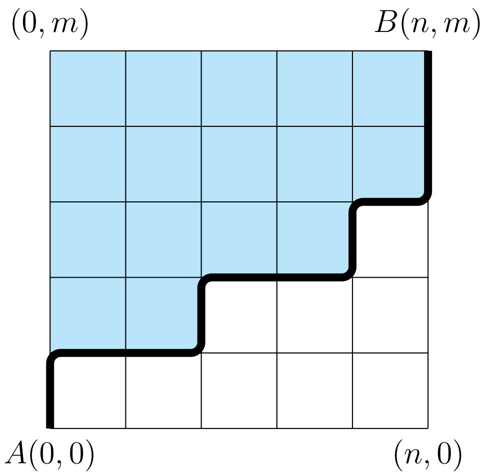


Figure 6.2.3: $m \times n$ Grid Walk Creating a Ferrers Diagram

Therefore, the number of paths, or the number of Ferrers diagrams, that can be formed is

$$\boxed{\binom{m+n}{n} = \binom{m+n}{m}}.$$

†

Let's look at a final example.

Proposition 6.2.6

The number of partitions of non-negative integer n that can be formed by k parts is given by

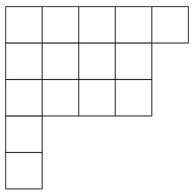
$$p(n,k) = p(n-k,k) + p(n-1,k-1).$$

Proof. Since there are no explicit formulas, an algebraic proof doesn't seem feasible. Instead, double counting may be applied by interpreting the Ferrers diagram.

Take an arbitrary partition. Consequently, the Ferrers diagram form of this partition may either have a singleton (a single block) on the bottom row or more than 1. For example, two partitions of 15 with a singleton last row and a non-singleton last row in the form of a Ferrers diagram are shown below.

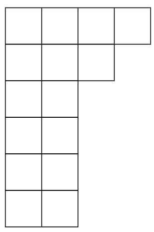
15=5+4+4+1+1, (5,4,4,1,1),

p(15,5): Partitioning 15 into 5 parts



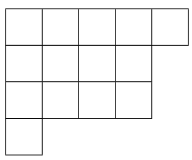
15=4+3+2+2+2+2, (4,3,2,2,2,2),

p(15,6): Partitioning 15 into 6 parts



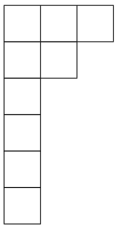
The LHS of the recurrence is a partition of n into k parts. The RHS can be described by the two cases previously mentioned.

The first case can be recursively defined by removing the last row, which results in the previous integer partition into one less part, $p(n-1,k-1)$ (each row is considered a part). For the partition of 15, this can be visualized.



This now represents the partition of 14 into 4 parts, $p(14,4)$, instead of 15 into 5 parts, $p(15,5)$. Therefore, we obtain $p(n-1,k-1)$ from this case, as shown on the RHS of the equation. The number of partitions of singleton cases for $p(n,k)$ has a bijection to the number of partitions for $p(n-1,k-1)$ because this process is reversible: for each "bijected" case of $p(n-1,k-1)$, adding a new bottom row containing a singleton block obtains the case from $p(n,k)$ that it pertains to in the bijection.

The second case can also be recursively defined by removing the first column, resulting in k less area, meaning the partitioned number will be k less. Note that no rows will be deleted since this is the case in which the last row will have more than 1 block, meaning the rows above it do not either, due to the definition of the Ferrers diagram and partitions being weakly decreasing while moving down the rows. Therefore, we have the same k parts that we are partitioning, equating the $p(n-k,k)$ on the RHS of the equation. This can be seen in the partition example of 15 with more than 1 block in the last row, which previously partitioned 15 into 6 parts:



This was $p(15,6)$ and it became $p(9,6)$, which satisfies $p(15-6,6)$.

This process is also reversible because no rows were altered so that there exists a bijection and the $p(n-k,k)$ cases can add a column to the left to obtain their corresponding partition of $p(n,k)$. Therefore, $p(n-k,k)$ counts in this case.

Adding the first and second cases results in all partitions, as partitions can only end with either 1 or more blocks, as the rules of Ferrers diagram state that there must be at least one block per row. The LHS counts all partitions of n into k parts, while the RHS counts partitions of the last row ending with a singleton and the partitions where the last row contains more than one block. □

A good visualization of this recurrence proof is demonstrated in this Youtube video by Mathematical Visual Proofs.

Exercises 6.2

Exercise 6.2.1

Draw the Ferrers diagram for the partition $(4,2,1)$.

Exercise 6.2.2

Draw the Ferrers diagram for the partition $(3,3,2,1)$.

Exercise 6.2.3

Draw the Ferrers diagram for the partition $(5, 3, 2)$.

Exercise 6.2.4

How many different Ferrers diagrams can you draw for a partition of 9?

Exercise 6.2.5

Draw a Ferrers diagram for the partition $(5, 4, 1)$ and then draw one for the conjugate partition.

Exercise 6.2.6

A standard Young tableau is a Ferrers diagram numbered from 1 to n, where n is the integer that is being partitioned. The top left corner starts with 1, and increases going to the right, as well as going down. Every number should only be used once.

Exercise 6.2.7

Find the Ferrers diagram for the conjugate of any partition of x, the number of distinct partitions of 10, into 4 parts.

Exercise 6.2.8

Use Ferrers graphs to show that the number of partitions of n is equal to the number of partitions of $2n$ that have exactly n summands.

Exercise 6.2.9

The number of partitions of n into k parts is equal to the number of partitions of n into parts the largest of which is k.

Exercise 6.2.10

The number of partitions of n into parts where no part occurs more than once is equal to the number of partitions of n where no two parts are consecutive integers.

6.3 Generating Function for Partitions

Recall that $p(n)$ represents the number of integer partitions of n. While there isn't a pretty closed formula for partitions, we can use a generating function approach to model these integer partitions.

Like many generating function problems, to find the generating function, it is often easier to break them into parts and model each one with its own generating function, then multiply every generating function to find the whole's generating function, which we will denote by $P(x)$. The general generating function form is given by the following expression, in which finding the coefficient of x^n will give our desired $p(n)$. This is the definition of generating functions.

$$P(x) = \sum_{n\geq 1} p(n)x^n$$

To visualize the compositions that can create the whole generating function, Ferrers diagram can be drawn.

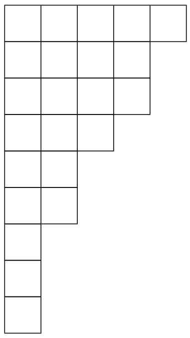


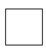
Figure 6.3.1: Example Ferrers Diagram Partition of 23

An integer n is naturally composed of parts such as those as rows shown in the Ferrers diagram above. Each part can be thought as some z-block "lego" pieces. For example, the first row of Figure 6.3.1 contains a 5-block piece, while there are two 4-block pieces in the next two columns, and so on.

Following this logic, partitions that can be formed for an integer n using solely each piece can be inspected as the small parts in order to find its generating function. This generating function would basically be the number of ways to create a partition using only y-block pieces. If the generating function of all pieces containing any amount of blocks is found, they can be simply multiplied to form the generating function of any partition using any block sized piece.

For a 1-block piece, the generating function can be written as

$$A(x) = (A_0, A_1, A_2, A_3, \dots) = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots.$$



Recall that the coefficients A_y represent the number of ways to do a task by using some items or whatever is presented by the problem. In this case, they represent the number of ways to form a partition of y using only 1's as parts.

So for A_0, the task is to find the number of ways a partition of 0 can be formed from one 1-block piece. The answer to that is 1, since by the Principle of Nothingness, there is only one way to do nothing.

For A_1, the number of partitions of 1 that can be formed from 1-block pieces is 1.

For A_2, the number of partitions of 2 that can be formed from only 1-block pieces is again, 1.

The pattern should be evident now: y 1-block pieces can each form one partition. Therefore, the generating function follows $(1, 1, 1, \dots)$.

Therefore, the generating function for using 1-block pieces is

$$\boxed{A(x) = 1 + x + x^2 + \dots}.$$

2-block pieces also follow the convention,

$$B(x) = (B_0, B_1, B_2, B_3, \dots) = B_0 + B_1 x + B_2 x^2 + B_3 x^3 + \dots.$$

B_0 represents the number of partitions of $y = 0$ that can be formed using one 2-block piece, which is just one partition. Only the partition of 0 can be formed, since there is one way to do nothing. Therefore, $B_0 = 1$.

B_1 represents the number of partitions of 1 able to be derived from only 2-block pieces. There are no partitions that can be created because no amount of 2-block pieces can be used to create 1 block. Therefore, $B_1 = 0$.

B_2 behaves similar to B_0. There is only one way to achieve an area of 2, or a partition of 2, using only 2-block pieces. That is just using one 2-block piece to create an area of 2, any more or any less would not work. Therefore, $B_2 = 1$.

There is a bijection between partitions of even, non-negative integers and some number of 2-block pieces, and therefore, using 2-block pieces inform that only one way every even partitions are able to be created. Therefore, $B(x) = (1, 0, 1, 0, 1, \dots) = 1 + 0x + x^2 + 0x^3 + \dots = \boxed{1 + x^2 + x^4 + \dots}$.

This pattern pertains to every n-sized piece, as every of those has one way to do nothing and one way to form areas of multiples of n due to a bijection. Therefore, 3-block pieces has a generating function of $C(x) = (1, 0, 0, 1, 0, 0, 1, \dots) = \boxed{1 + x^3 + x^6 + x^9 + \dots}$.

This goes on for any n. Multiplying all generating functions of the generating functions of ways to create partitions using each n sized part obtains

$$P(x) = A(x) \times B(x) \times C(x) \times \dots =$$
$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots$$

Utilizing the geometric sum formula (introduced in chapter 2), this generating function can be rewritten. Again, the geometric sum formula states:

$$\frac{r^n - 1}{r - 1} = \sum_{k=1}^{n} r^{k-1} = 1 + r + r^2 + ... + r^{n-1}$$

In the case of $(1 + x + x^2 + \dots)$, $r = x$ and $n = \infty$. This yields the following, keeping in mind that the x in generating functions is a number that follows the inequality $-1 < x < 1$:

$$\frac{x^\infty - 1}{x - 1} = \frac{-1}{x - 1} = \boxed{\frac{1}{1 - x}}.$$

$(1 + x^2 + x^4 + \dots)$ can also be rewritten using this formula, this time knowing that each time $r = x^2$, or even faster, knowing that x can be directly substituted for x^2 from $(1 + x + x^2 + \dots)$ to obtain $(1 + x^2 + x^4 + \dots)$, this can be done to the geometric sum formula that was just obtained. Therefore,

$$(1 + x^2 + x^4 + \dots) = \boxed{\frac{1}{1 - x^2}}.$$

Following the same logic, $(1 + x^3 + x^6 + \dots)$ is equal to $\boxed{\frac{1}{1 - x^3}}$.

Therefore, $P(x)$ can be rewritten as

$$P(x) = \frac{1}{1 - x} \times \frac{1}{1 - x^2} \times \frac{1}{1 - x^3} \times \dots.$$

And an easier way to write this would be

$$P(x) = \prod_{m \geq 1} \frac{1}{1 - x^m}.$$

Definition 6.3.1

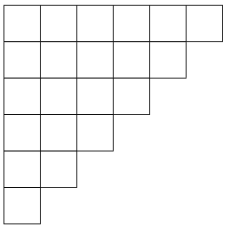
The generating function of a partition is given by

$$P(x) = \prod_{m=1}^{\infty} \frac{1}{1 - x^m}.$$

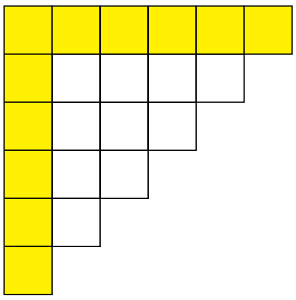
Proposition 6.3.2

There exists a bijection between self-conjugate partitions of n with partitions of n with distinct, odd parts. That is, the number of self-conjugate partitions are equal to the number of distinct, odd partitions for any non-negative integer partition of n ($\lambda \vdash n$).

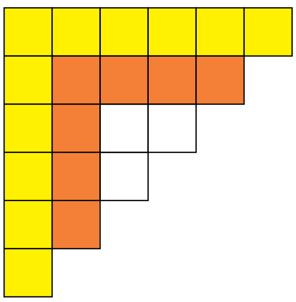
Proof. Consider any self-conjugate partitions $\lambda \vdash n, \lambda \in X$, where X is the set of self-conjugate partitions. Then, consider a Ferrers diagram representation, denoted by $\text{FD}(\lambda)$, of one such partition.



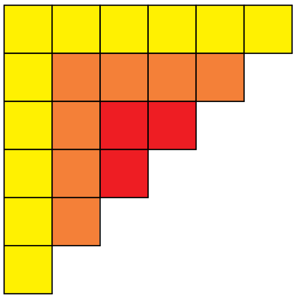
Color the first row and first column of $\text{FD}(\lambda)$ some color c_1:



Then, color uncolored squares in the second row and second column a different color c_2:

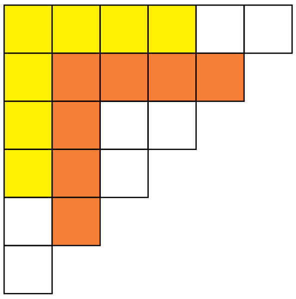


Perform this operation until all blocks of $\text{FD}(\lambda)$ are colored.

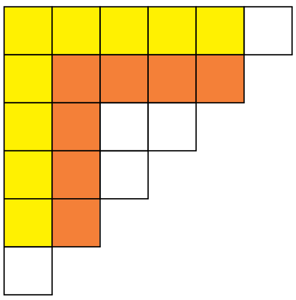


Then, notice that since this arbitrary figure is self-conjugate, any row i must have equal amount of blocks as column i. Let the number of blocks in a column or row i be equal to m. This means that the color c_i has some $2m-1$ blocks by summing the row and column and subtracting the one overlapping block (or alternatively, by PIE). The parity of $2m$ is even because m is an integer and anything multiplied by 2 is even, meaning that the parity of $2m-1$ must, consequently, be odd. Therefore, the numbers of blocks covered by each coloring c_i must be odd.

Next, there are no pairwise equivalent number of blocks by colorings, otherwise it would not be a valid Ferrers diagram as shown below (where there are 7 yellow and 7 orange blocks):



Clearly, the least difference a different, adjacent coloring can have in terms of blocks is 2, where both colorings stop at the same row and column (orange has 7 blocks while yellow has 9):



Therefore, counting the number of blocks by each coloring (parts) would biject the partition of n into k parts into a partition of odd, distinct parts using the same amount of blocks. A bijection occurs because adding or removing two blocks would change the partition, meaning each n being partitioned shares only one partition of odd, distinct parts that corresponds to only one self-conjugate partition. $\square$

Theorem 6.3.3 (Sylvester's theorem)
Sylvester's theorem states that the number of partitions of an integer n into odd parts equals the number of partitions of n into distinct parts

By the definition of generating functions (that equal events have equal generating functions), this proof enables the discovery of a generating function for self-conjugate partitions since the generating function for partitions with odd, distinct parts can be easily determined.

The generating function of partitions with odd, distinct parts ($P_{\text{d, odd}}(x)$) can be found by breaking it down into steps.

First, having only odd parts means that instead of having the

$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots,$$

only the generating function of odd parts will be multiplied, resulting in the following expression:

$$(1 + x + x^2 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^5 + x^{10} + \dots)\dots$$

Now the distinct parts feature needs to be accounted for. Understand that each generating function such as $(1 + x + x^2 + \dots)$ represents the generating function for using only some fixed block pieces. Since parts must all be distinct, the maximum number of parts of the same number of blocked pieces that can be used is once. The minimum is of course zero, using it no times. Therefore, $(1+x+x^2+\dots)$ would become $(1 + x)$ since the first coefficient represents the number of ways to reach a partition of 0 (i.e. using no 1-block pieces), the second coefficient is the number of ways to reach a partition of 1 (i.e. using one 1-block piece), and any other coefficients like the third one would be impossible to create since it is a partition of 2 and it is only possible to reach a partition of 2 or more using 2 or more 1-block pieces, which would result in a violation of our requirements.

Therefore, only the first two terms of each generating function for odd parts will be valid. This is

$$P_{\text{d, odd}}(x) = (1 + x)(1 + x^3)(1 + x^5)(1 + x^7)\dots$$

Another way to express this is

$$P_{\text{d, odd}}(x) = \prod_{k=1}^{\infty} \left(1 + x^{2k-1}\right) = \prod_{k \text{ odd}} \left(1 + x^k\right).$$

Definition 6.3.4 (Generating Function for Self-Conjugate Partitions)

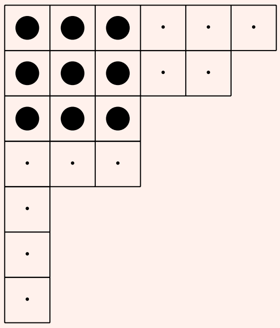
The generating function for self-conjugate partitions (and partitions with distinct, odd parts) is given by

$$\begin{aligned}
P_{\text{d, odd}}(x) &= (1+x)(1+x^3)(1+x^5)(1+x^7)\dots \\
&= \prod_{k=1}^{\infty}\left(1+x^{2k-1}\right) \\
&= \prod_{k \text{ odd}}\left(1+x^k\right).
\end{aligned}$$

Exercises 6.3

Exercise 6.3.1

A *Durfee square* of a partition λ is the largest square that fits in its top-left corner. For example, the Durfee square of $\lambda = (6,5,3,3,1,1,1)$ is of size 3:



Use Durfee squares to show that the generating function for partitions can be written

$$\prod_{i=1}^{\infty}\frac{1}{1-q^i} = \sum_{i=0}^{\infty}\frac{q^{i^2}}{(1-q)^2(1-q^2)^2\cdots(1-q^i)^2}.$$

Exercise 6.3.2

Find a simple expression for

$$\prod_{k=0}^{\infty}(1+x^{2^k}) = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$$

Exercise 6.3.3

Prove that the number of partitions of n into odd parts greater than 1 equals the number of partitions of n into distinct parts that are not powers of 2.

Exercise 6.3.4

Show that the number of partitions of n in which no part occurs exactly once is equal to the number of partitions of n in which no part is congruent to 1 or 5 modulo 6.

Exercise 6.3.5

Show that the number of partitions of n in which no part occurs exactly once is equal to the number of partitions of n in which no part is congruent to 1 or 5 modulo 6.

Exercise 6.3.6

Let $f(n)$ be the number of partitions of n, where each part occurs at most twice. Find a set S of positive integers, such that $f(n)$ is equal to the number of partitions of n where each part is an element of S.

Exercise 6.3.7

Let

$$\prod_{n=1}^{1996}(1+nx^{3^n}) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m},$$

where $a_1, a_2, \ldots, a_m$ are nonzero and $k_1 < k_2 < \cdots < k_m$. Find a_{1996}.

Exercise 6.3.8

Prove that

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$

Exercise 6.3.9

Find the generating function for partitons into odd parts, with no other restrictions on repetition. Use this to find the number of partitions of 25 into odd parts.

Exercise 6.3.10

What is the generating function for the number of partitions of an integer in which each part is even?

Exercise 6.3.11

Write down the generating function for the number of ways to partition an integer into parts of size no more than m, each used an odd number of times. Write down the generating function for the number of partitions of an integer into parts of size no more than m, each used an even number of times. Use these two generating functions to get a relationship between the two sequences for which you wrote down the generating functions.

Exercise 6.3.12

Use the fact that

$$\frac{1-q^{2i}}{1-q^i} = 1 + q^i$$

and the generating function for the number of partitions of an integer into distinct parts to show how the number of partitions of an integer k into distinct parts is related to the number of partitions of an integer k into odd parts.

Exercise 6.3.13

Prove using generating functions that the number of partitions of n such that adjacent parts differ by at least 2 is equal to the number of partitions of n such that each partition is either $1 \mod 5$ or $4 \mod 5$.

Exercise 6.3.14

Recall Proposition 6.3.2. Prove the other way (similar to the one shown in the lesson) that partitions of distinct, odd parts is equal to self-conjugate partitions by starting with partitions of odd, distinct parts.

Exercise 6.3.15

Show that

$$|\{\lambda \vdash n : \text{even } \#\text{even parts}\}| - |\{\lambda \vdash n : \text{odd } \#\text{even parts}\}|$$

$$= \left|\{\lambda \vdash n : \lambda = \lambda^\top\}\right| \text{(self-conjugate partitions)}$$

Exercise 6.3.16

Show that the number of partitions of a positive integer n where no summand appears more than twice equals the number of partitions of n where no summand is divisible by 3.

Exercise 6.3.17

Consider the partition function $p(n)$, which denotes the number of partitions of a positive integer n. Euler's Pentagonal Number Theorem provides a fascinating way to compute $p(n)$ using a recurrence relation involving earlier values of the partition function.

Prove that the partition function $p(n)$ satisfies the following recurrence relation:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

where the indices k are given by generalized pentagonal numbers $\frac{m(3m-1)}{2}$ for $m = \pm 1, \pm 2, \pm 3, \ldots$.

Hint: Start by considering the generating function for the partition function, defined as:

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

Derive a recurrence relation for $p(n)$ by expanding and analyzing this generating function.

Exercise 6.3.18

Verify the recurrence relation of the previous theorem once for specific values of n (e.g., $n = 5, 10, 12, 15$).

6.4 Set Partitions and Bell Numbers

Partitions of a set are somewhat similar to integer partitions. Let set S contain three elements, where $S = \{1, 2, 3\}$. A partition of this set splits it into subsets whose union is set S and all numbers appear exactly once. Thus set partitions generally follows these two rules:

(1) The partition is a set P containing a collection of non-empty subsets of set S.
(2) Every element in set S is in exactly one element of the partition set P.

In the case where $S = \{1, 2, 3\}$, valid partitions P would include $P = \{\{2\}, \{1, 3\}\}$ or $P = \{\{1\}, \{2\}, \{3\}\}$. Invalid partitions include $\{\{1, 2\}, \{\}\}$ or $\{\{1, 2\}, \{2, 3\}\}$. Note the sets in the partitions are just labels and the only thing that matters is the set size, as this applies for any set of the same size, even if they have different elements than the ones used in these examples.

Definition 6.4.1 (Partition of a Set)
A partition P of a set S is a collection of non-empty subsets of S such that every element of S is in exactly one of the subsets (of S) apart of P.
In other words, a partition of a set is a set of non-empty disjoint subsets of S whose union is equal to S.

Set partitions aim to break sets into subsets (or "groups") where order does not matter and repetition is not allowed. Imagine a rectangle box representing set S containing scattered numbers as shown in the figure below.

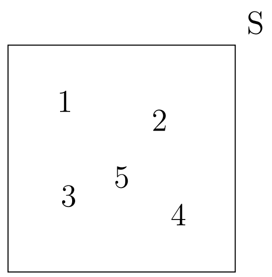


Figure 6.4.1: Numbers in Rectangular Box S

This set can then be partitioned into four groups or "subsets" $\{\{1\}, \{2\},$ $\{4\}, \{3, 5\}\}$. This is shown in Figure 6.4.2 below.

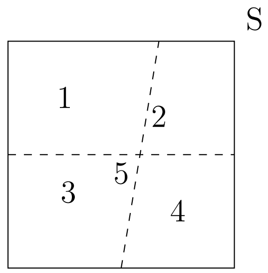


Figure 6.4.2: Grouped Numbers in Rectangular Box S

It is important to note that the separation of a n-set called S into n distinct subsets whose union is S is called the partition of S into *singletons*. This terminology was used for singular blocks on the Ferrers diagram, but they also apply to singular subsets that the singular blocks corresponded to. The correct way to refer to the subsets of the partition of a 3-set $S = \{1, 2, 3\}$ for the case of $\{\{1\}, \{2, 3\}\}$ is the partition into a singleton and a 2-set.

In partitions of integers, the notation for the number of partitions for an integer was expressed as $p(n)$. For set partitions, the notation for the number of possible partitions is given by Bell Numbers, $B(n)$.

Definition 6.4.2 (Bell Numbers)

The Bell number B_n is defined as the number of partitions of a set of n elements. Formally, for a given positive integer n, the n-th Bell number B_n is given by:

$$B_n = |\{\mathcal{P} \mid \mathcal{P} \text{ is a partition of } \{1, 2, \ldots, n\}\}|$$

where $\mathcal{P}$ denotes a partition of the set $\{1, 2, \ldots, n\}$, and the vertical bars denote the cardinality of the set of all such partitions.

Example 6.4.3

Find the first 4 Bell numbers, with the first one being B_0.

Solution. B_0 is the number of partitions that a set with zero elements can have. By the Principle of Nothingness, there is only one way to do nothing, and therefore $B_0 = 1$.

B_1 counts the number of partitions for a set with one element. We can use numbers as examples of elements, so given $S = \{1\}$, there is only one partition, which is the subset that is equal to the set, $P = \{\{1\}\}$.

B_2 is the number of partitions with a two-element set, which we can denote an example with numbers $S = \{1, 2\}$. This has two possible partitions: $P = \{\{1\}, \{2\}\}$ and $P = \{\{1, 2\}\}$. Therefore, $B_2 = 2$.

B_3 is represented by the number of partitions able to be formed from the set $S = \{1, 2, 3\}$. This can be $P = \{\{1, 2\}, \{3\}\}$, $P = \{\{1, 2, 3\}\}$, $P = \{\{1, 3\}, \{2\}\}$, $\{\{2, 3\}, \{1\}\}$, and $P = \{\{1\}, \{2\}, \{3\}\}$. This means that $B_3 = 5$. †

This keeps going and these Bell numbers are in fact the same as the partitions of integers.

There is something called the Bell triangle that can help compute these values way quicker. The Bell triangle incorporates a recurrence formula similar to Proposition 6.2.6, which states that $P(n,k) = P(n-1,k-1)+P(n-k,k)$. This similar recurrence formula will be evident in Definition 6.4.4 on the next page. The first few rows of the Bell triangle can be seen below.

1					
1	2				
2	3	5			
5	7	10	15		
15	20	27	37	52	
52	67	87	114	151	203

Figure 6.4.3: Bell Triangle

Definition 6.4.4 (The Bell Triangle)

The Bell triangle is a triangular array of numbers that generates Bell numbers.

Let $T(n,k)$ denote the entry of the n-th row and k-th column where $n \geq k$. The Bell triangle can then be constructed under these rules:

1. The first element on the first row and column is $B_0 = 1$ ($T(0,0)$).

2. When $k = 0$ and $n > 0$, $T(n,0)$ is equal to the last entry on previous row $n-1$.

3. When $k > 1$ and $n > 0$, $T(n,k)$ is obtained by adding the entry above it to the entry to $T(n,k)$'s left. That is, $T(n,k) = T(n-1,k-1) + T(n,k-1)$.

4. $T(n,k) = 0$ for $k > n$.

The n-th Bell number, B_n, can be found by the leftmost entry (on the n-th row) of the Bell triangle. Basically, $B_n = T(n,0)$.

The $T(n,k) = T(n-1,k-1)+T(n,k-1)$ recurrence relation will be further discussed in the next section. Also note that Figure 6.4.3's diagonal also contains Bell numbers, but starting with B_0 instead of B_1.

The four Bell numbers that was calculated in the previous example solution can be seen in the first four elements of the first column. In addition, it becomes clear that $B_4 = 15$ and $B_5 = 52$.

Notice how the Hockey-Stick identity is also prevalent in the Bell Triangle, but instead of diagonal sticks, they are horizontal. Additionally, the end of the stick is one number longer, with the end perpendicular to the stick consisting of the first item of the current row and of the last row. The stick is formed by all entries of the row that is two rows above the current one. This cool feature is created by the addition process of creating the Bell triangle, similar to Pascal's.

A few instances of this can be seen in Figure 6.4.3. For example, the first row sums to 1, and the hockey stick has an end that is two parts long, meaning that we have the handle part being $1 + 0 + 0 + 0 + \ldots$ on the first row and the first item of the second row, 1, is to be added to this equaling the number at the end of the hockey stick, the first number of the third row (2). Indeed, $1 + 0 + 0 + \cdots + 1 = 2$. A more clear example of this is the sum of the second row plus the first item of the third row is equal to the first item of the fourth row, creating another hockey stick. That is, $2 + 1 + 2 = 5$. A visual of this can be seen bolded below.

$$\begin{array}{llllll}
1 \\
\mathbf{1} & \mathbf{2} \\
\mathbf{2} & 3 & 5 \\
\mathbf{5} & 7 & 10 & 15 \\
15 & 20 & 27 & 37 & 52 \\
52 & 67 & 87 & 114 & 151 & 203
\end{array}$$

Another example of this is the sum of the third row added to the first item of the fourth row is equal to the first item of the fifth row. That is, $5 + 3 + 2 + 5 = 15$. This sequence is bolded below.

$$\begin{array}{llllll}
1 \\
1 & 2 \\
\mathbf{2} & \mathbf{3} & \mathbf{5} \\
\mathbf{5} & 7 & 10 & 15 \\
\mathbf{15} & 20 & 27 & 37 & 52 \\
52 & 67 & 87 & 114 & 151 & 203
\end{array}$$

The last example that can be seen (since this one lists only a few rows) is the sum of the fourth row added to the first entry of the fifth row is equal to the first entry of the sixth row ($5 + 7 + 10 + 15 + 15 = 52$). This is again shown below in bold.

$$\begin{array}{llllll}
1 \\
1 & 2 \\
2 & 3 & 5 \\
\mathbf{5} & \mathbf{7} & \mathbf{10} & \mathbf{15} \\
\mathbf{15} & 20 & 27 & 37 & 52 \\
\mathbf{52} & 67 & 87 & 114 & 151 & 203
\end{array}$$

Example 6.4.5

Find all 15 partitions of a set with four elements, following that the Bell number $B_4 = 15$.

Solution. Keep in mind, there cannot exist a repetition in a set due to its natural property, meaning all items must be distinct by definition. Let set $S = \{1, 2, 3, 4\}$.

Starting with P having one subset, there is only one possible partition:

$$P = \{\{1, 2, 3, 4\}\}.$$

Looking for two subsets in the partition, the following displays 7 such partitions.

$$P = \{\{1, 2\}, \{3, 4\}\},$$

$$P = \{\{1, 3\}, \{2, 4\}\},$$
$$P = \{\{1, 4\}, \{2, 3\}\}.$$
$$P = \{\{1\}, \{2, 3, 4\}\}.$$
$$P = \{\{2\}, \{1, 3, 4\}\}.$$
$$P = \{\{3\}, \{1, 2, 4\}\}.$$
$$P = \{\{4\}, \{1, 2, 3\}\}.$$

Moving onto three partitions, we obtain six partitions.

$$P = \{\{1\}, \{2\}, \{3, 4\}\}$$

$$P = \{\{1\}, \{3\}, \{2, 4\}\}$$
$$P = \{\{1\}, \{4\}, \{2, 3\}\}$$
$$P = \{\{2\}, \{3\}, \{1, 4\}\}$$
$$P = \{\{3\}, \{4\}, \{1, 2\}\}$$
$$P = \{\{2\}, \{4\}, \{1, 3\}\}$$

Finally, separating the set into four subsets is results in only one possible partition where each number is placed into their own subset.

$$P = \{\{1\}, \{2\}, \{3\}, \{4\}\}$$

Therefore, there are $B_4 = 1 + 7 + 6 + 1 = \boxed{15}$ partitions, confirming the Bell number given by the Bell triangle. †

One may notice that during the calculation of each partition for parts in the previous problem, binomial coefficients appear to be prominent in counting those partitions. For example, partitioning into two parts had two instances: one where both subsets had 2 elements and another where one subset had one element and the other had 3.

The first instance follows choosing two items from a set of four elements to be placed into the first subset (the other two for the second subset would be automatically chosen since they are the only ones left after the first two are picked). Then one would conclude that there are $\binom{4}{2} = 6$ possibilities. However, this is incorrect since it does not account for the second subset accounting for another instance for every first subset chosen. The second subset would always contain a different combination of two elements. This is due to the unordered property of sets, and in this case, the set P is causing such results. Therefore, there are $\binom{4}{2} \div 2 = 6 \div 2 = \boxed{3}$ such combinations, which can be seen in the solution.

The second instance is naturally just choosing one element for the first subset and the other three elements left would be automatically chosen into the second subset. This time, the second subset would not create another instance as they are different sizes. Therefore, there are just $\binom{4}{1} = \boxed{4}$ such possible partitions, as shown in the solution.

That is why 7 was obtained (because 3+4=7), as found through hard-listing all partitions.

There was another prevalent use of binomial coefficients to count these partitions in the same solution for when they were separated into three subsets. In this case, there are four elements to be separated into three groups (subsets). By the Pigeonhole Principle Part 1, there must be at least (and in this case, exactly) one group with two elements, as there are more elements than groups to not "overfill". Therefore, there are two subsets with one element each and the third subset has two elements.

It follows that choosing elements in either the third subset alone or only the first two would result in the rest of the subsets being chosen. For example, if the first two subsets were chosen, then the third subset has to be the last two elements remaining. If the third subset is chosen, then the last two elements must fall within the last two subsets, and since sets are unordered, both cases count as one partition. Therefore, there are $\binom{4}{2} = \boxed{6}$ such partitions.

This agrees with the 6 partitions listed and along with partitioning into one subset (choosing four elements for that one subset, $\binom{4}{4} = \boxed{1}$) and partitioning into four subsets (choosing one element for four subsets, $\binom{4}{4} = \boxed{1}$), 7+6+1+1=15 partitions are, again, obtained. Note that partitioning into four subsets can also be thought of as choosing three elements and the last one will be chosen ($\binom{4}{3} = 4$), but there are four same-sized subsets, meaning that all orderings will be considered one partition due to the unordered nature of subsets (1 partition).

In addition to the presence of binomial coefficients in counting these partitions, the Bell triangle is also analogous and quite similar to Pascal's triangle, meaning that perhaps Bell numbers have some kind of relationship with binomial coefficients.

Indeed, there is a recurrence relation for Bell numbers that is constructed utilizing binomial coefficients. Keep in mind that Bell numbers, similar to partitions, do not have a pretty explicit formula for calculating them.

Theorem 6.4.6 (Recurrence Relation for Bell Numbers)
A recurrence relation of Bell numbers can be defined by a sum containing binomial coefficients:

$$\begin{aligned} B_{n+1} &= \binom{n}{0}B_0 + \binom{n}{1}B_1 + \cdots + \binom{n}{n}B_n \\ &= \sum_{k=0}^{n}\binom{n}{k}B_k. \end{aligned}$$

Example 6.4.7
Prove Theorem 6.4.6 by applying the double counting proof technique.

Proof. Let "equation m" be

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k$$

and let set $S = \{1, 2, \ldots, n, n+1\}$. By inspection, the LHS counts the number of partitions that can be formed from set S containing $n+1$ distinct objects.

Like many identities, double counting the more complex side splits the simpler side into many cases. Let partition P contain a subset A_1 with the element $n+1$ and a cardinality equal to $k+1$ ($|A_1| = k+1$). This means that there are k objects other than $n+1$. Since A_1 must contain $n+1$, the number of ways to choose subset A_1 is equal to the number of ways to choose the k other objects since A_1 is already fixed to the set. In other words, this is choosing k elements from a set of n elements (from set S that excludes the already chosen element $n+1$). $\binom{n}{k}$ represents this action.

If one of the subsets of the partition P is already chosen and the number of possible combinations of elements it can have as a $k+1$-sized set fixed with $n+1$ is $\binom{n}{k}$, then multiplying this to the number of ways the other subsets can be formed will result in the number of ways to partition with A_1 already choosing the element $n+1$. The number of ways to choose "other" subsets is also the number of ways to partition the $n+1$ items after $k+1$ has been chosen, which is the same as the number of ways to partition $n+1-(k+1) = n-k$ distinct items, not including the element $n+1$. This is given by the Bell number B_{n-k}.

This gives $\binom{n}{k}B_{n-k}$ many ways to partition a set of $n+1$ elements into subsets where the first one is fixed with the element $n+1$ and has $k+1$ elements. Since all possible values of $k+1$ needs to be accounted for in order to model all partitions of $n+1$, this is just $\sum_{k+1=1}^{n+1}\binom{n}{k}B_{n-k}$. The summand goes from 1 to $n+1$ because the least amount of elements subset A_1 can have is 1, meaning there is only the forced-chosen element $n+1$ and no other elements are chosen. The highest cardinality of A_1 is $n+1$, where every element of S is in this set, including $n+1$. $k+1$ ranging from 1 to $n+1$ is the same as k summing from 0 to n, thus allowing a rewritten form $\sum_{k=0}^{n}\binom{n}{k}B_{n-k}$. This formula counts the partitions of $n+1$ elements of set S by first choosing the element $n+1$ into subset A_1.

Finally, recall the symmetrical property given by Pascal's triangle from Chapter 2 that states $\binom{n}{k} = \binom{n}{n-k}$. Therefore, by subsituting $\binom{n}{k}$ with $\binom{n}{n-k}$,

$$\sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} B_{n-k}.$$

The expanded form is

$$\sum_{k=0}^{n} \binom{n}{n-k} B_{n-k} = \binom{n}{n} B_n + \binom{n}{n-1} B_{n-1} + \cdots + \binom{n}{1} B_1 + \binom{n}{0} B_0.$$

And therefore, this can be rewritten as

$$\sum_{k=0}^{n} \binom{n}{n-k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

This is the RHS of equation m. Thus, the LHS gives the Bell number for $n+1$, counting the partitions that can be formed from a set S with $n+1$ elements. The RHS counts the number of partitions able to be formed by starting by finding the number of partitions that can be formed by a specific subset A_1 that contains the element $n+1$, then finding the number of partitions of that form the rest of the subsets, and then finally summing over all possible sizes of set A_1. $\square$

Recall the first few Bell numbers are 1, 1, 2, 5, 15, 52. We previously derived $B_4 = 15$ by raw listing all the partitions. Let's now use this theorem to verify results, given that $(B_0, B_1) = (1, 1)$.

B_2 can be rewritten as B_{1+1}, where $n = 1$. Using Theorem 6.4.6,

$$\begin{aligned} B_{1+1} &= \sum_{k=0}^{1} \binom{1}{k} B_k \\ &= \binom{1}{0} B_0 + \binom{1}{1} B_1 \\ &= B_0 + B_1 \\ &= \boxed{2} \end{aligned}$$

Similarly, $B_3 = B_{2+1}$, where $n = 2$. Theorem 6.4.6 gives

$$\begin{aligned} B_{2+1} &= \sum_{k=0}^{2} \binom{2}{k} B_k \\ &= \binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \\ &= B_0 + 2(B_1) + B_2 \end{aligned}$$

$$
\begin{aligned}
&= (1) + 2(1) + (2) \\
&= \boxed{5}
\end{aligned}
$$

B_4 is equal to B_{3+1}, using that $n = 3$. Applying the Theorem 6.4.6,

$$
\begin{aligned}
B_{3+1} &= \sum_{k=0}^{3} \binom{3}{k} B_k \\
&= \binom{3}{0} B_0 + \binom{3}{1} B_1 + \binom{3}{2} B_2 + \binom{3}{3} B_3 \\
&= B_0 + 3(B_1) + 3(B_2) + B_3 \\
&= (1) + 3(1) + 3(2) + (5) \\
&= \boxed{15}
\end{aligned}
$$

Finally, $B_5 = B_{4+1}$. This means $n = 4$, and Theorem 6.4.6 gives

$$
\begin{aligned}
B_{4+1} &= \sum_{k=0}^{4} \binom{4}{k} B_k \\
&= \binom{4}{0} B_0 + \binom{4}{1} B_1 + \binom{4}{2} B_2 + \binom{4}{3} B_3 + \binom{4}{4} B_4 \\
&= B_0 + 4(B_1) + 6(B_2) + 4(B_3) + B_4 \\
&= (1) + 4(1) + 6(2) + 4(5) + (15) \\
&= \boxed{52}
\end{aligned}
$$

Thus, this theorem confirms the first few Bell numbers and provides an easier way to find the rest.

Exercises 6.4

Exercise 6.4.1

Find the Bell number B_7 using the Bell triangle.

Exercise 6.4.2

Find the Bell number B_7 using Theorem 6.4.6.

Exercise 6.4.3

Given a set of 10 elements, how many ways can you partition this set into two subsets, each containing exactly 5 elements?

Exercise 6.4.4

Find the number of ways to partition a set of 7 elements into exactly 3 non-empty subsets such that one subset has exactly 4 elements.

Exercise 6.4.5

Consider the set $\{1, 2, 3, 4, 5\}$. How many partitions of this set have at least one subset of size exactly 2?

Exercise 6.4.6

Show that if $\{A_1, A_2, \dots, A_k\}$ is a partition of $\{1, 2, \dots, n\}$, then there is a unique equivalence relation $\sim$ whose equivalence classes are $\{A_1, A_2, \dots, A_k\}$.

Exercise 6.4.7

Suppose n is a square-free number, that is, no number m^2 divides n; put another way, square-free numbers are products of distinct prime factors, that is, $n = p_1 p_2 \cdots p_k$, where each p_i is prime and no two prime factors are equal.

Find the number of factorizations of n. For example, $30 = 2 \cdot 3 \cdot 5$, and the factorizations of 30 are $30, 6 \cdot 5, 10 \cdot 3, 2 \cdot 15$, and $3 \cdot 2 \cdot 5$. Note we count 30 alone as a factorization, though in some sense a trivial factorization.

Exercise 6.4.8

Select any 10 numbers from the set $[1, 100]$. Prove that it is possible to partition these numbers into two non-empty subsets such that the sum of the numbers in each subset is identical.

Exercise 6.4.9

The rhyme scheme of a stanza of poetry indicates which lines rhyme. This is usually expressed in the form ABAB, meaning the first and third lines of a four-line stanza rhyme, as do the second and fourth, or ABCB, meaning only lines two and four rhyme, and so on. A limerick is a five-line poem with a rhyming scheme AABBA.

How many different rhyme schemes are possible for an n-line stanza? To avoid duplicate patterns, we only allow a new letter into the pattern when all previous letters have been used to the left of the new one. For example, ACBA is not allowed, since when C is placed in position 2, B has not been used to the left. This is the same rhyme scheme as ABCA, which is allowed.

Exercise 6.4.10

Let A_n be the number of partitions of $\{1, 2, \ldots, n+1\}$ in which no consecutive integers are in the same part of the partition. For example, when $n = 3$ these partitions are $\{\{1\}, \{2\}, \{3, 4\}\}$, $\{\{1\}, \{2, 4\}, \{3\}\}$, $\{\{1, 3\}, \{2\}, \{4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$, so $A_3 = 5$.

Let $A(n, k)$ be the number of partitions of $\{1, 2, \ldots, n+1\}$ into exactly k parts, in which no consecutive integers are in the same part of the partition. Thus

$$A_n = \sum_{k=2}^{n+1} A(n, k).$$

Find a recurrence for $A(n, k)$ and then show that $A_n = B_n$.

Exercise 6.4.11

Bell numbers grow rapidly, and the asymptotic approximation for large n is given by:

$$B_n \sim \frac{1}{\sqrt{n}} \left(\frac{n}{\ln n}\right)^n$$

Use this approximation to estimate B_{15} and compare it with the actual value (using the Bell triangle or the theorem).

Exercise 6.4.12

Use Theorem 6.4.6 to derive an exponential generating function for Bell numbers.

Exercise 6.4.13

Consider the exponential generating function for Bell numbers from the previous problem. Use this generating function to derive B_7.

6.5 Stirling Numbers of the Second Kind

Recall splitting a partition of integer n into k parts, denoted by $p_k(n)$. For set partitions, this can be called a k-partition of partitioning a set into k subsets.

Definition 6.5.1 (Partitioning a Set into k Subsets)

A k-partition partitions a set into k subsets. For an n-set, the following notation is used:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \text{\# of } k\text{-partitions of an } n\text{-set},$$

where an n-set is a set containing n (distinct) elements.

Below is an altered version of Exercise 6.4.3, and a problem of listing all partitions of 4 was previously discussed. This example again breaks it down and will use a binomial coefficient reasoning as was done before, but in a slightly different manner, to introduce an interesting topic.

Example 6.5.2

Given a set of 6 elements, how many total ways can you partition this set into subsets?

Namely, how many partitions into one subset are there? Two subsets each containing exactly 3 elements? One containing 2 and the other 4? One containing 1 and the other 5? 3 subsets? 4? 5? 6?

Solution. Let this 6-set be set $S = \{1, 2, 3, 4, 5, 6\}$. The number of ways to partition it into one subset is if it is an equal set, a subset equal to set S. $P = \{\{1, 2, 3, 4, 5, 6\}\}$. This counts $\boxed{1}$ partition.

Partitioning into two subsets can be split into three cases based on subset size, as mentioned in the question: two 3-sets, one 2-set and a 4-set, and one 1-set and a 5-set.

Starting with case 1, with two 3-sets, the goal is to group a set of 6 numbers, or objects, into one set, or group, of three objects. This will automatically choose the other set. This is $\binom{6}{3}$. But since the second subset is equal to the first subset in terms of size, the second subset will create another distinct combination that the first subset accounted for by doing $\binom{6}{3}$. Since order doesn't matter for sets, we have over-counted 2!. Therefore, there are $\binom{6}{3} \div 2 = 20 \div 2 = \boxed{10}$.

The next case is where there is a 2-set and a 4-set. This is choosing the 6 numbers from set S to place into a 2-set, which would automatically choose the 4-set. This is simply $\binom{6}{2}$, or equivalently $\binom{6}{4}$ due to Pascal's symmetry identity where I can also choose the items for 4-set that will automatically choose the 2-set. So we have $\binom{6}{2} = \boxed{15}$.

The third case involves a 1-set and a 5-set. This again is just $\binom{6}{1} = \binom{6}{5} = \boxed{6}$.

For three subsets, the sizes can vary from (2,2,2), (3,2,1), and (4,1,1). Notice how these are informed by weakly decreasing order when partitioning into the three sizes.

For (2,2,2), there are $\binom{6}{2} \times \binom{4}{2} = 15 \times 6 = 90$ ways to choose the first 2-set. However, there are three indistinguishable objects, so we divide by the number of arrangements, 3!. So we have $90 \div 6 = \boxed{15}$ ways to do this.

For (3,2,1), we can choose the 1-set and then the 2-set: $\binom{6}{1} \times \binom{5}{2} = 6 \times 10 = \boxed{60}$ ways to do this. There is no indistinguishable sets since they are of different sizes, which is the prime difference between sets in partitions.

For (4,1,1), we choose the 1-set then the other 1-set: $\binom{6}{1} \times \binom{5}{1} = 6 \times 5 = 30$. Since the two 1-sets are indistinguishable, we must divide by 2!. This is $30 \div 2 = \boxed{15}$.

Four subsets include the following combinations: (2,2,1,1) and (3,1,1,1). (2,2,1,1) can have the two 1-sets chosen first and then one of the 2-sets chosen and the last 2-set would be chosen automatically. This is $\binom{6}{1} \times \binom{5}{1} \times \binom{4}{2} = 6 \times 5 \times 6 = 180$. But the 2-sets and the 1-sets are indistinguishable, so we divide by $2! \times 2! = 4$. Therefore, this counts $180 \div 4 = \boxed{45}$ partitions. (3,1,1,1) can just have the three 1-sets chosen, so this is $\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} = 6 \times 5 \times 4 = 120$. Then, we divide by the rearrangements possible, which is 3! from the 1-sets. This is $120 \div 3! = 20$ such partitions. A faster way would have been choosing the 3-set directly, which avoids any indistinguishability: $\binom{6}{3} = \boxed{20}$.

Next, the number of subsets with 5 elements can be separated into cases. 5 subsets can be broken down into subsets of sizes (2,1,1,1,1) and...that's it, any other ordering would be a rearrangement of that. This is $\binom{6}{4} = \binom{6}{2} = \boxed{15}$ by choosing the 2-set first, leaving the four 1-sets to be automatically chosen with the remaining 4 elements. We do not need to divide since they are indistinguishable, and they do not account for additional partitions of 2-set, since the 2-set size is different. We could have also done the $\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} \times \binom{3}{1} = 6 \times 5 \times 4 \times 3 = 360$, which we would then divide by the amount of times these 1-sets can be rearrnaged, which is 4!. And $360 \div 4! = 360 \div 24 = 15$.

Lastly, there is only one way to arrange 6 elements into 6 groups, by placing them each into 1 element subsets. $\binom{6}{5} \div \binom{6}{5} = \boxed{1}$ describes choosing the first five 1-sets and then the last 1-set is indistinguishable from the other 1-sets so we divide by 6.

Here, we find that $1 + (10 + 15 + 6) + (15 + 60 + 15) + (45 + 20) + 1 = \boxed{203}$ partitions, which is indeed the Bell number B_6. †

- What patterns do you notice on these partitions?

Observation: for a combination of 2 subsets where $|S| = n$, we inspect every combination of subset sizes a_1, a_2 of $a_1 + a_2 = n$, where $1 \leq a_1 < a_2 \leq n - 1$. For any cases where $a_1 = a_2$, we divide by $\binom{2}{1}$. Similarly, for any 3-sized subset, we solve $a + a_2 + a_3 = n$ where $1 \leq a_1 < a_2 < a_3 \leq n - 1$. If any two are similar, we divide by the rearrangements 2!, and if any three are similar, we divide by 3!, and so on. If we use the long route of choosing all except one subset, then we account for both situations that you would need to divide by the arrangements: (1) there are multiple same-sized subsets and also (2) when you choose a subset of some size and what remains is another subset of the same size.

Therefore, we first choose a combination of x subsets by taking the "long route" of finding the number of ways to choose all subsets except the last one. In repeated sizes cases, we divide by $m!$ arrangements where m is the number of equivalent a_i's (multiple would be multiplied, for example if there are two 2's and three 3's, it is $\binom{3}{2} \times \binom{2}{1}$). Then, all possible partitions from each combination of $a_1 + a_2 + \cdots + a_x = n$ are summed (where $1 \leq a_1 < a_2 < \cdots < a_x \leq n - 1$) to obtain the total partitions of the n-set.

- Can you generalize a formula for an n-set for perhaps partitions into 1 subset $\left\{ n \atop 1 \right\}$, only 2-sets $\left\{ n \atop 2 \right\}$, only $n-1$-sets $\left\{ n \atop n-1 \right\}$, and only n-sets $\left\{ n \atop n \right\}$?

Starting with the easy ones, $\left\{ n \atop 1 \right\}$ and $\left\{ n \atop n \right\}$ are obviously equal to 1 partition each. There's only one way to distribute a set into a subset that is equal to that set, and there is also only one way to distribute a n-set into n non-empty sets.

Proposition 6.5.3

$$\left\{ n \atop 2 \right\} = 2^{n-1} - 1.$$

Proof. Suppose you have a set S with n elements, where $S = [n]$. You want to find the number of ways to assign each element of S into two subsets, A_1 and A_2. For each element, there are two choices: they either go to A_1 or A_2, since the union must be S, each element of S needs to go to one of the sets. This is 2^n ways.

However, partitions require non-empty subsets, in which this counts that. Therefore, by complementary counting, we achieve our answer by subtracting the two cases that do not work, which are when all elements go to set A_1 ($|A_1|= n, |A_2|= 0$) or the other way. This is $2^n - 2$ ways to achieve that.

Remember when we had sizes of $(3, 3)$ for a partition of 6 into two subsets and we had to divide by the multiplicity of 3 in order to account for the rearrangements possible (since choosing for the first 3-set chooses another distinct combination of a 3-set for the second pairing)? Therefore we divide by $2! = 2$, obtaining $\frac{2^n-2}{2} = 2^{n-1} - 1$. $\square$

Proposition 6.5.4

$$\left\{ n \atop n-1 \right\} = \binom{n}{2}.$$

Proof. Suppose there is a set S such that $|S|= n$ and $S = [n]$. To partition this into $n-1$ subsets, the Pigeonhole Principle Part 1 states that there must be one group with more than one element, since there are n elements and $n-1$ groups.

Remember when we partitioned 6 into $(4, 1, 1)$? The number of ways to do this was either $\binom{6}{1} \times \binom{5}{1} \div 2! = 15$ or $\binom{6}{4}$ ways. The first method is done by choosing every subset except one and then dividing by the rearrangements possible since the 1-sets are indistinguishable. For $n-2$ 1-sets and one 2-set, there would be $n-2$ indistinguishable 1-sets to be divided after finding the number of ways to choose those sets excluding the last 2-set that will be automatically chosen:

$$\binom{n}{1} \times \binom{n-1}{1} \times \ldots \times \binom{3}{1} \div (n-2)! = n \times n-1 \times \ldots \times 4 \times 3 \div (n-2)!$$
$$= \frac{n!}{(2 \times 1)!\,(n-2)!}$$
$$= \binom{n}{2}.$$

The second way would have resulted in the same formula, where we just have to choose the 2-set and the rest will be chosen since they are indistinguishable and any rearrangement would be one partition due to the set being unordered. The number of ways to choose a subset of size 2 from a set of n is equal to $\binom{n}{2}$. □

Example 6.5.5

Using these formulas, formulate an explicit formula for the partition of any n-set,

$$\left\{ {n \atop k} \right\}.$$

Summing a finite number of formulae results in the desired partition. However, finding a general formula for each of these cases would be impossible since n is an arbitrary integer. Therefore, establishing a solid relationship occurring in each case with n and k would generalize this for all n, k. So with our new understanding of finding partitions into a specific amount of parts both numerically and generally, let's use an overarching method that is common to counting partitions of any n into any k-subset.

Let set $S = [n]$, where $[n]$ is the set of the first n natural numbers, $\{1, 2, \ldots, n\}$. $\left\{ {n \atop k} \right\}$ counts the number of ways of splitting set S into k non-empty disjoint subsets. This is the same thing as arranging n distinguishable balls into k indistinguishable boxes.

Remember when we often used the Pigeonhole Principle Part 1 to say that one of the subsets must have more than x items? Well, this idea can be applied here too, where $n \geq k$. Recall surjections, a function in which every element in the codomain C has at least one preimage ("element") in the domain D. In other words, $\forall c \in C, \exists d \in D$ s.t. $f(a) = b$.

The number of surjections from $[n]$ to $[k]$ counts the relationship between the balls and boxes and makes sure every k is mapped onto that satisfies the non-empty subsets requirement. If that is divided by $k!$, which would make sure it all arrangements of an ordering is counted as one partition, then it essentially counts the desired partition of arranging n distinguishable balls into k indistinguishable boxes.

$$\left\{ {n \atop k} \right\} = \#\text{ of surjections } [n] \to [k] \div k!$$

The number of surjections has an explicit formula that can be proved using complementary counting and the Principle of Exclusion-Inclusion. Note that the notation ":=" refers to "is defined as."

Proposition 6.5.6

The number of surjections S is given by

$$S = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n.$$

Proof. For $1 \leq i \leq m$, let

$$B_i := \{f : [n] \to [m] \mid i \notin \operatorname{Im}(f)\},$$

where $[n] := \{1, 2, \ldots, n\}$ and $[m] := \{1, 2, \ldots, m\}$. Note that $\operatorname{Im}(f)$ describes the image of f, which is the collection of the codomain consisting of the m that are mapped onto by n. B_i essentially counts the number of elements in m that are not mapped onto by any of some k chosen elements of $[n]$.

By complementary counting, the number of surjections from $[n]$ to $[m]$ is given by the cardinality of the complement of

$$\bigcup_{i=1}^{m} B_i$$

in the set of all mappings from $[n]$ to $[m]$. The number of all possible mappings from n to m is equal to m^n because each element of $[n]$ has m elements that they can map to, which is

$$\underbrace{m \times m \times m \times \ldots \times m}_{m^n}.$$

Therefore, we have

$$S = \left| \left(\bigcup_{i=1}^{m} B_i \right)^{\complement} \right| = m^n - \left| \bigcup_{i=1}^{m} B_i \right|.$$

Next, we replace

$$\left| \bigcup_{i=1}^{m} B_i \right|$$

with the following using the converse of the Principle of Inclusion Exclusion:

$$\sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq m} \left| \bigcap_{l=1}^{k} B_{i_l} \right|.$$

The expression

$$\sum_{1 \leq i_1 < \cdots < i_k \leq m} \left| \bigcap_{l=1}^{k} B_{i_l} \right|$$

can be simplified. The cardinality of the intersections of B_{i_l} from $l = 1$ to k counts the number of m elements that are not mapped onto by any of the k chosen elements $i_1, i_2, \ldots, i_k$ in the function $f : [n] \to [m]$. Since there are k elements being chosen in the image of f (in other words, mapped onto for every n), there are $(m-k)$ elements not mapped onto by those chosen elements. This is $(m-k)^n$ many instances that follow that rule. The summation sums over all combinations of values of some i_j that can, and since each iteration only changes values of the i_j terms, it does not affect $(m-k)^n$.

Therefore, this can be simplified by multiplying the number of iterations by the summand. The number of iterations is just the number of ways to choose the k i_j elements from the set of m objects since $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. This is just $\binom{m}{k}$.

Therefore,

$$\sum_{1\leq i_1<\cdots<i_k\leq m} \left| \bigcap_{l=1}^{k} B_{i_l} \right| = \binom{m}{k}(m-k)^n.$$

Substituting back, we obtain that

$$\sum_{k=1}^{m}(-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq m} \left| \bigcap_{l=1}^{k} B_{i_l} \right| = \sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k}(m-k)^n.$$

Substituting this into the original equation and simplifying the negative to multiply with the $(-1)^{k-1}$ in every term when expanded leaves us with

$$S = m^n + \sum_{k=1}^{m}(-1)^k\binom{m}{k}(m-k)^n.$$

Finally, m^n can be combined into the summation as the case where $k = 0$ since $(-1)^0\binom{m}{0}(m-0)^n = m^n$:

$$S = \boxed{\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n},$$

which gives the number of surjections from a set of size n to a set of size m. □

Theorem 6.5.7 (Counting Surjections)

The number of surjections S mapping a n-set to a m-set is

$$S = \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n.$$

Now that we have found an explicit formula for the number of surjections from a set to another, we can substitute back into the original expression to find $\left\{ {n \atop k} \right\}$, but we substitute k with i and m with k, since we were partitioning n into m and now are doing it into k subsets:

$$\left\{ {n \atop k} \right\} = \#\text{ of surjections } [n] \to [k] \div k!$$

$$= \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n \div k!$$
$$= \boxed{\frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}.$$

Theorem 6.5.8 (Stirling Numbers of the Second Kind)

Suppose set S contains n elements. The number of partitions of set S into k non-empty subsets is given by the explicit formula of Stirling numbers of the second kind, $S_2(n,k)$:

$$S_2(n,k) = \left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.$$

Example 6.5.9

Using Stirling numbers of the second kind, find the number of partitions of set $S = [6]$ into k parts (subsets), for $1 \le k \le 6$. Then, use this to solve for the total number of partitions of set S (the Bell number B_6).

Solution. $\left\{ {6 \atop 1} \right\}$:

$$\begin{aligned} \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n &= \frac{1}{1!} \sum_{i=0}^{1} (-1)^i \binom{1}{i} (1-i)^6 \\ &= \sum_{i=0}^{1} (-1)^i \binom{1}{i} (1-i)^6 \\ &= (-1)^0 \binom{1}{0} (1-0)^6 + (-1)^1 \binom{1}{1} (1-1)^6 \\ &= (-1)^0 \binom{1}{0} (1-0)^6 \\ &= \boxed{1}. \end{aligned}$$

$\left\{ \begin{matrix} 6 \\ 2 \end{matrix} \right\}$:

$$\begin{aligned}
\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n &= \frac{1}{2!}\sum_{i=0}^{2}(-1)^i\binom{2}{i}(2-i)^6 \\
&= \frac{1}{2}\sum_{i=0}^{2}(-1)^i\binom{2}{i}(2-i)^6 \\
&= \frac{1}{2}\left[(-1)^0\binom{2}{0}(2-0)^6 + (-1)^1\binom{2}{1}(2-1)^6\right. \\
&\quad \left. + (-1)^2\binom{2}{2}(2-2)^6\right] \\
&= \frac{1}{2}\left[1\cdot 64 - 2\cdot 1 + 1\cdot 0\right] \\
&= \boxed{31}.
\end{aligned}$$

$\left\{ \begin{matrix} 6 \\ 3 \end{matrix} \right\}$:

$$\begin{aligned}
\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n &= \frac{1}{3!}\sum_{i=0}^{3}(-1)^i\binom{3}{i}(3-i)^6 \\
&= \frac{1}{6}\sum_{i=0}^{3}(-1)^i\binom{3}{i}(3-i)^6 \\
&= \frac{1}{6}\left[1\cdot 729 - 3\cdot 64 + 3\cdot 1 - 1\cdot 0\right] \\
&= \boxed{90}.
\end{aligned}$$

$\left\{ \begin{matrix} 6 \\ 4 \end{matrix} \right\}$:

$$\begin{aligned}
\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n &= \frac{1}{4!}\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^6 \\
&= \frac{1}{24}\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^6 \\
&= \frac{1}{24}\left[1\cdot 4096 - 4\cdot 729 + 6\cdot 64 - 4\cdot 1 + 1\cdot 0\right] \\
&= \boxed{65}.
\end{aligned}$$

$\begin{Bmatrix} 6 \\ 5 \end{Bmatrix}$:

$$\begin{aligned}
\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n &= \frac{1}{5!}\sum_{i=0}^{5}(-1)^i\binom{5}{i}(5-i)^6 \\
&= \frac{1}{120}\sum_{i=0}^{5}(-1)^i\binom{5}{i}(5-i)^6 \\
&= \frac{1}{120}\left[1\cdot 15625 - 5\cdot 4096 + 10\cdot 729 - 10\cdot 64 + 5\cdot 1\right] \\
&\quad - 1\cdot 0 \\
&= \boxed{15}.
\end{aligned}$$

$\begin{Bmatrix} 6 \\ 6 \end{Bmatrix}$:

$$\begin{aligned}
\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n &= \frac{1}{6!}\sum_{i=0}^{6}(-1)^i\binom{6}{i}(6-i)^6 \\
&= \frac{1}{720}\sum_{i=0}^{6}(-1)^i\binom{6}{i}(6-i)^6 \\
&= \frac{1}{720}\left[1\cdot 46656 - 6\cdot 15625 + 15\cdot 4096 - 20\cdot 729\right] \\
&\quad + 15\cdot 64 - 6\cdot 1 + 1\cdot 0 \\
&= \boxed{1}.
\end{aligned}$$

Therefore, by adding all the partitions for each part, $1+31+90+65+15+1=203$, we obtain that $B_6 = \boxed{203}$. †

This method achieved the same result by using casework, as done in Example 6.5.2, but in a much more solid and quicker manner. In addition to the casework solution, the last item on the diagonal of the Bell triangle illustrated in Figure 6.4.3 also confirms our answer.

Below is Table 6.5.1, a table compilation of Stirling numbers of the second kind for n and k for $1 \leq n \leq 10, 1 \leq k \leq 10$, and $k \leq n$. The values for this table were extracted from the sequence A008277 in the OEIS website.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	0	1	1								
3	0	1	3	1							
4	0	1	7	6	1						
5	0	1	15	25	10	1					
6	0	1	31	90	65	15	1				
7	0	1	63	301	350	140	21	1			
8	0	1	127	966	1701	1050	266	28	1		
9	0	1	255	3025	7770	6951	2646	462	36	1	
10	0	1	511	9330	34105	42525	22827	5880	750	45	1

Table 6.5.1: Triangular Array of Stirling Numbers of the Second

Notice that the sum of each row results in a Bell number, as also done in Example 6.5.9.

Theorem 6.5.10

Since the Stirling numbers of the second kind counts the partitions of a n-set into k subsets, the sum over all possible numbers of subsets k gives the total number of partitions of a set with n members, or namely, the Bell number B_n.

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

Lastly, let's quickly look at another formula involving Stirling numbers of the second kind, but instead of an explicit formula, it is a recurrence formula.

Theorem 6.5.11 (Stirling Numbers Recurrence)

The Stirling numbers of the second kind satisfy the recurrence relation:

$$S_2(n,k) = k \cdot S_2(n-1,k) + S_2(n-1,k-1)$$

with the initial conditions:

$$S_2(0,0) = 1, \quad S_2(n,0) = 0 \text{ for } n > 0, \text{ and } S_2(0,k) = 0 \text{ for } k > 0.$$

Note that $S_2(0,k) = 0$ because partitions of a set 0 would result in empty subset(s), a violation of the definition of set partitions. Therefore, there are no such partitions.

We will again use the double counting proof technique.

Proof. Observe the partition of a set containing n objects (specifically, $[n] = \{1, 2, \ldots, n\}$) into k subsets. This describes the LHS of the equation.

Similar to the construction of the Bell number recurrence relation, one object is focused on. This partition will either contain a singleton subset with the element n or it will not.

For the cases that it does contain n in a singleton subset, the number of ways to partition the rest of the $n-1$ objects into $k-1$ other subsets is equal to

$$S_2(n-1, k-1).$$

In the cases that the element n shares a subset with other element(s), the number of ways to choose the other $n-1$ objects, first, into k subsets is $S_2(n-1, k)$.

For the cases that it was singleton, it would be swapping its now-empty subset with some other subset, which counts as the same partition due to sets being unordered. Since object n does not hold its own subset now, changing the subset that it is apart of informs a different partition, as there will be some objects left in the subset it was already apart of, and therefore the subset that this partition is moving into does not "swap" back to the other subset (or else it will be a singleton again).

Therefore, there are k possible subsets n can be chosen to, obtaining

$$k \cdot S_2(n-1, k)$$

ways to partition this set into k parts when element n is not apart of a singleton subset.

Summing the two obtained expressions results in the desired RHS. Therefore, this equation is double counted to find that the LHS partitioning n elements into k parts is the same as partitioning that same set into the same amount of subsets but by breaking it down into two cases where a specific element n is either apart of a singleton subset or not. □

Refer back to Table 6.5.1. This recurrence relation that was just proven can be used to construct that table. Let's try to verify one of the items on the table using this recurrence relation.

Example 6.5.12

Find $S_2(4, 2)$ using only Theorem 6.5.11.

Solution. Using Theorem 6.5.11, we obtain the following:

Base cases based on initial conditions identities stated in the theorem:

$$\begin{aligned} S_2(2,2) &= 1 \quad \text{(base case)} \\ S_2(2,1) &= 1 \quad \text{(base case)} \\ S_2(2,0) &= 0 \quad \text{(base case)} \end{aligned}$$

Then, obtain the following using the recursive formula:

$$S_2(4,2) = k \cdot S_2(n-1,k) + S_2(n-1,k-1)$$

$$\begin{aligned}
&= 2 \cdot S_2(4-1,2) + S_2(4-1,2-1)\\
&= 2 \cdot S_2(3,2) + S_2(3,1)\\
&= 2 \cdot (2 \cdot S_2(2,2) + S_2(2,1)) + (S_2(2,1) + S_2(2,0))\\
&= 2 \cdot (2 \cdot 1 + 1) + (1 + 0)\\
&= 2 \cdot 3 + 1\\
&= 6 + 1\\
&= \boxed{7}.
\end{aligned}$$

†

From Table 6.5.1, the $S_2(4,2)$ is indeed equal to 7. Thus, we have proven an example showcasing this theorem.

Try verifying some more values on the table on your own and see how it informs the patterns of the triangular array, similar to the Bell and Pascal's triangle.

Exercises 6.5

Exercise 6.5.1

Find the number of ways to partition a set of 8 elements into exactly 3 subsets where each subset has at least 2 elements.

Exercise 6.5.2

The (3,1,1) notation for a partition of 5, for example, is similar to a sequence called the type vector sequence containing $(a_1, a_2, \ldots, a_k)$ where each a_i is the number of subsets of size i. For the example of partitioning [5] into 3 subsets, a valid vector sequence would be $(2,0,1,0,0)$. Note that $\sum_{i=1}^{k} a_i = n$ and $\sum_{i=1}^{k} ia_i = k$.

Find the number of ways to partition a set of 7 elements [7] into exactly 4 non-empty subsets. List all type vectors and the number of partitions for each one.

Exercise 6.5.3

Find the number of partitions of a set of 9 elements into exactly 3 subsets where each subset has at least 1 element.

Exercise 6.5.4

Imagine that you are going to serve n kids ice cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavors get mixed, show that the number of ways we can give out the cones using all k flavors.

Exercise 6.5.5

In how many ways may the caterer distribute the nine sandwiches into three identical bags so that each bag gets exactly three?

Exercise 6.5.6

For Exercise 6.5.5, how many ways could the sandwiches be distributed if the bags were distinct instead of identical?

Exercise 6.5.7

Consider a set K with k elements. In how many ways can we assign these k elements into n distinct groups such that the first group has j_1 elements, the second group has j_2 elements, and so on, with the nth group having j_n elements? This is analogous to partitioning a set into subsets but now we label the subsets. This number is given by the *multinomial coefficient*, a topic that could be used for solving the previous problem, given by:

$$\binom{k}{j_1, j_2, \ldots, j_n}$$

Exercise 6.5.8

Using the concept of multinomial coefficients, explain how you would compute the number of ways to distribute the elements of a k-element set K into n distinct groups.

Exercise 6.5.9

Now, consider if each group from the previous problem corresponds to a distinct label. How would the multinomial coefficient help you compute the number of such functions where each element from set K is assigned to one of the n labels?

Exercise 6.5.10

Relate the idea of expanding the power of a multinomial expression $(x_1 + x_2 + \cdots + x_n)^k$ with the concept of multinomial coefficients. How does this relate to the previous activities where we assigned elements to distinct groups?

Exercise 6.5.11

Prove the following theorem. (Consider using a generating functions approach.)

$$\begin{aligned} x^k &= S(k,1)\binom{x}{1}1! + S(k,2)\binom{x}{2}2! + \cdots + S(k,k)\binom{x}{k}k! \\ &= \sum_{m=1}^{k} S(k,m)\binom{x}{m}m! \end{aligned}$$

Exercise 6.5.12

How many subsets $\{a, b, c\}$ are there of $\{2, 3, 4, \ldots\}$ such that $abc = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$?

Exercise 6.5.13

Let $S = \{2, 3, 4, \ldots\}$. How many ordered triples (a, b, c) are there in $S \times S \times S$ such that $abc = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$?

Index

Bibliography

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2. "Path Photos Transparent HQ PNG Download |FreePNGImg." *FreePNGImg*, 2017, freepngimg.com/png/28024-path-photos. Accessed Sept. 2024.

This book consists of competition math, proof, and author-made exercises and problems with unique solutions tailored to **prepare high-school students for mathematical competitions** as well as those looking to self-study an **introduction to applied combinatorics and discrete math** courses in college.

This book teaches useful problem-solving skills, strategies, and shortcuts, and the **necessary skills to construct proofs** that are not only essential to combinatorial identities but also to writing and understanding mathematics as a whole. With **thoroughly walked-through example problems**, this book provides an easily comprehensible platform for learning, enrichment, and practice.

The intended use of this book is a teacher's manual or a self-study handbook for dedicated students, and **includes many exercise questions available at all difficulty levels** at the end of each chapter for extra practice.

The author **provides many important clearly-labeled definitions, corollaries, lemmas, propositions, and theorems** within each chapter. Extra features include an index and table of contents for easier navigation, and **easily-accessible lecture videos** as alternative lessons adjusted for preference.